Consider a massless real scalar field $\phi(t,x)$ in $1+1$ dimensions. Suppose $0 < x < L$ with Dirichlet boundary conditions at the endpoints of the interval.

$$\phi(t, x = 0) = \phi(t, x = L) = 0$$

The action for the field is

$$S = \int_{-\infty}^{\infty} dt \int_0^L dx \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - \epsilon_0 \right] .$$

Here ϵ_0 is a constant representing a uniform ground state energy density. Usually one ignores ϵ_0, since it has no effect on the dynamics, but it will be important in the sequel.

1. As in (10.33), expand

$$\phi(t,x) = \sum_{n=1}^{\infty} \phi_n(t) \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} .$$

Plug this into the action and show that you get an infinite collection of harmonic oscillators. Identify the frequencies ω_n.

2. Quantize each oscillator in the usual way, setting

$$\begin{aligned} \phi_n &= \frac{1}{\sqrt{2\omega_n}} (a_n + a_n^\dagger) \\ \pi_n = \dot{\phi}_n &= -i\sqrt{\frac{\omega_n}{2}} (a_n - a_n^\dagger) . \end{aligned}$$

Here ϕ_n is the oscillator coordinate, π_n is its conjugate momentum, and a_n, $a_n^\dagger$ obey the usual commutation relations: $[a_n, a_m^\dagger] = \delta_{nm}$. Argue that the Hamiltonian for the field is

$$H = L\epsilon_0 + \sum_{n=1}^{\infty} \omega_n \left(a_n^\dagger a_n + \frac{1}{2} \right) .$$

Note that, unlike Zwiebach (10.60), we're keeping track of the zero-point energy!

3. Argue that the ground state energy of the field is

$$E_0 = L\epsilon_0 + \sum_{n=1}^{\infty} \frac{n\pi}{2L} .$$

This is clearly divergent. To define it introduce a cutoff scale Λ (a quantity with dimensions of energy or inverse length) and set

$$E_0(\Lambda) \equiv L\epsilon_0(\Lambda) + \sum_{n=1}^{\infty} \frac{n\pi}{2L} e^{-n\pi/\Lambda L} . \qquad (1)$$

You can think of Λ as an ad hoc, short-distance modification to the theory. Note that we are allowing the parameter ϵ_0 to depend on Λ. But – the crucial point which makes the whole thing well-defined – we do *not* allow ϵ_0 to depend on any other quantities such as the size of the system L.

4. Compute the sum in (1). Expand your answer for large Λ and show that as $\Lambda \to \infty$

$$E_0(\Lambda) = \underbrace{L\epsilon_0(\Lambda) + \text{const.}\, L\Lambda^2}_{L\epsilon_{\text{phys}}} + \underbrace{\text{finite stuff}}_{E_{\text{Casimir}}} .$$

The first two terms define the physical or renormalized ground state energy density ϵ_{phys} that the system would have in infinite volume ($L \to \infty$). One conventionally chooses $\epsilon_0(\Lambda)$ to set $\epsilon_{\text{phys}} = 0$.[1] The remaining ground state energy that is left over in the $\Lambda \to \infty$ limit is known as the Casimir energy.

5. On p. 221 Zwiebach follows a remarkable procedure: he says

$$\sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12}$$

and therefore

$$\sum_{n=1}^{\infty} \frac{n\pi}{2L} = -\frac{\pi}{24L} .$$

What do you think?

[1]More generally, one could set ϵ_{phys} to any constant value.