Gauge symmetry — Strings course
Fall '07

It's sometimes convenient to formulate a theory in terms of redundant degrees of freedom. Such theories are known as gauge theories. Electromagnetism is usually formulated as a gauge theory, as is the relativistic point particle we encountered last time. But non-relativistic theories can also have gauge symmetries, as I hope the following calculations make clear.

1. Consider a non-relativistic particle of mass m moving in a potential $V(\mathbf{x})$. The action for a trajectory $\mathbf{x}(t)$ is

$$S = \int dt \left(\frac{1}{2} m \left| \frac{d\mathbf{x}}{dt} \right|^2 - V(\mathbf{x}) \right) .$$

   Work out the canonical momentum $\mathbf{p} = \frac{\partial L}{\partial(d\mathbf{x}/dt)}$, the canonical Hamiltonian $H = \mathbf{p} \cdot \frac{d\mathbf{x}}{dt} - L$, and the Euler-Lagrange equations of motion.

2. Suppose we wanted to parametrize our particle's trajectory, not by time t, but by some arbitrary parameter λ: $\mathbf{x} = \mathbf{x}(\lambda)$. We can do this by introducing an additional degree of freedom $t(\lambda)$ and promoting the action to the following reparametrization-invariant form.

$$S = \int d\lambda \left(\frac{m}{2\frac{dt}{d\lambda}} \left| \frac{d\mathbf{x}}{d\lambda} \right|^2 - \frac{dt}{d\lambda} V(\mathbf{x}) \right)$$

   This action is invariant under arbitrary reparametrizations $\lambda \to \tilde{\lambda} = \tilde{\lambda}(\lambda)$. Under an infinitesimal reparametrization $\lambda \to \lambda + \epsilon(\lambda)$ we have

$$\delta t = \epsilon \frac{dt}{d\lambda} \qquad\qquad \delta \mathbf{x} = \epsilon \frac{d\mathbf{x}}{d\lambda} .$$

   This means $t(\lambda)$ is a redundant or gauge degree of freedom; we could always choose $t(\lambda) = \lambda$ and recover our previous action.

3. Suppose we stick with our reparametrization-invariant form of the action. Work out the canonical momenta

$$p_t = \frac{\partial L}{\partial(dt/d\lambda)} \qquad \mathbf{p}_x = \frac{\partial L}{\partial(d\mathbf{x}/d\lambda)} ,$$

the canonical Hamiltonian

$$H = p_t \frac{dt}{d\lambda} + \mathbf{p}_x \cdot \frac{d\mathbf{x}}{d\lambda} - L\,,$$

and the Euler-Lagrange equations of motion.

4. You might think there's something fishy about introducing $t(\lambda)$ as an independent degree of freedom, and indeed there is: show that the canonical momenta are not independent, but rather are related by $p_t = -\frac{1}{2m}|\mathbf{p}_x|^2 - V$. This is a common property of a gauge theory: since t isn't a true degree of freedom, it doesn't have an independent canonical momentum. Rather the canonical momenta satisfy some kind of constraint equation.

5. Show that the Euler-Lagrange equation of motion for t implies that p_t is conserved. What's the physical interpretation of this conservation law?

6. The vanishing Hamiltonian is a common feature of reparametrization-invariant theories. To see the physical interpretation, note that the Hamiltonian generates infinitesimal reparametrizations under which λ shifts by a constant, and any two configurations related by such shifts are physically equivalent.

7. What do you get if you repeat steps 1 – 6 starting from the relativistic Lagrangian $L = -mc\sqrt{c^2 - \left|\frac{d\mathbf{x}}{dt}\right|^2}$ for a trajectory $\mathbf{x}(t)$?

Moral of the story: in non-relativistic mechanics it's a bit silly to introduce $t(\lambda)$ as an independent degree of freedom. But for a relativistic particle it's useful since it lets you characterize the particle's trajectory in terms of a Lorentz 4-vector $x^\mu(\lambda)$. Electromagnetism works in a similar way. One could describe electromagnetism in terms of two degrees of freedom (the two polarizations of a plane wave). But it's convenient to introduce redundant degrees of freedom and describe electromagnetism in terms of a vector potential A_μ, since that formulation makes Lorentz invariance manifest.