Path integrals in quantum mechanics Phys V3500/G8099
handout #1

References: there's a nice discussion of this material in the first chapter of L.S. Schulman, *Techniques and applications of path integration*.

Path integrals in quantum mechanics

Consider the standard 1-d quantum mechanics problem of a particle moving in a potential (but let's set $\hbar = 1$).

$$H = \frac{p^2}{2m} + V \equiv T + V$$

The basic quantity of interest is the time evolution operator

$$U(t_1, t_0) = e^{-iH(t_1 - t_0)}$$

or rather its matrix elements in a position basis

$$G(x_1, t_1 | x_0, t_0) = \langle x_1 | e^{-iH(t_1 - t_0)} | x_0 \rangle \,.$$

Knowing G is tantamount to solving the Schrodinger equation since

$$\begin{aligned}
\psi(x_1, t_1) &= \langle x_1 | \psi(t_1) \rangle \\
&= \langle x_1 | e^{-iH(t_1 - t_0)} | \psi(t_0) \rangle \\
&= \int dx_0 \, \langle x_1 | e^{-iH(t_1 - t_0)} | x_0 \rangle \langle x_0 | \psi(t_0) \rangle \\
&= \int dx_0 \, G(x_1, t_1 | x_0, t_0) \psi(x_0, t_0)
\end{aligned}$$

Let's begin by studying time evolution over very short intervals, setting $t_1 = t_0 + \epsilon$. Then we have

$$\begin{aligned}
\langle x_1 | e^{-iH\epsilon} | x_0 \rangle &\approx \langle x_1 | \mathbb{1} - i\epsilon (T + V) | x_0 \rangle \\
&\approx \langle x_1 | e^{-i\epsilon T} e^{-i\epsilon V} | x_0 \rangle
\end{aligned}$$

where the error in the last line is $\mathcal{O}(\epsilon^2)$. I'll assume the error is negligible; for a more careful discussion see the "Trotter product formula" in Schulman. Inserting a complete set of momentum eigenstates

$$\begin{aligned}
\langle x_1 | e^{-iH\epsilon} | x_0 \rangle &\approx \int dp \, \langle x_1 | e^{-i\epsilon T} | p \rangle \langle p | e^{-i\epsilon V} | x_0 \rangle \\
&= \int dp \, e^{-i\epsilon p^2/2m} e^{-i\epsilon V(x_0)} \frac{1}{2\pi} e^{ip(x_1 - x_0)}
\end{aligned}$$

Completing the square in the exponent

$$\langle x_1 | e^{-iH\epsilon} | x_0 \rangle \approx \int \frac{dp}{2\pi} \, e^{-\frac{i\epsilon}{2m}(p - \frac{m}{\epsilon}(x_1 - x_0))^2} e^{\frac{im}{2\epsilon}(x_1 - x_0)^2} e^{-i\epsilon V(x_0)}$$

Shifting variables of integration and performing the Gaussian integral

$$\begin{aligned}
\langle x_1 | e^{-iH\epsilon} | x_0 \rangle &\approx \int \frac{dp}{2\pi} \, e^{-\frac{i\epsilon}{2m}p^2} e^{\frac{im}{2\epsilon}(x_1 - x_0)^2} e^{-i\epsilon V(x_0)} \\
&= \frac{1}{2\pi} \left(\frac{2\pi m}{i\epsilon} \right)^{1/2} e^{\frac{im}{2\epsilon}(x_1 - x_0)^2} e^{-i\epsilon V(x_0)} \\
&= \left(\frac{m}{2\pi i \epsilon} \right)^{1/2} e^{i\epsilon \left[\frac{1}{2} m \left(\frac{x_1 - x_0}{\epsilon} \right)^2 - V(x_0) \right]}
\end{aligned}$$

The quantity in square brackets is very intriguing. Suppose we tried to make sense of the action

$$S = \int_{t_0}^{t_1} dt \, \frac{1}{2} m \dot{x}^2 - V(x)$$

for paths connecting (x_0, t_0) to $(x_1, t_1 = t_0 + \epsilon)$. Adopting a finite-difference approximation for the particle velocity, it seems reasonable to say that

$$\int_{t_0}^{t_1} dt \, \frac{1}{2} m \dot{x}^2 \approx \epsilon \cdot \frac{1}{2} m \left(\frac{x_1 - x_0}{\epsilon} \right)^2 .$$

Also adopting a Riemann-sum definition of $\int dt \, V$, with a single term in the Riemann sum, it seems reasonable to say that

$$\int_{t_0}^{t_1} dt \, V \approx \epsilon V(x_0) \, .$$

This means the quantity in square brackets can be thought of as the classical Lagrangian, and the exponent can be thought of as i times the classical action!

So far this has all been for evolution over infinitesimal time intervals. But extending these results to finite time intervals is easy. All we have to do is insert lots of complete sets of position eigenstates.

$$\begin{aligned} \langle x_f|e^{-iH(t_f-t_i)}|x_i\rangle &= \int dx_1\cdots dx_{N-1}\,\langle x_N|e^{-iH(t_f-t_i)/N}|x_{N-1}\rangle\langle x_{N-1}|e^{-iH(t_f-t_i)/N}|x_{N-2}\rangle\langle x_{N-2}| \\ &\qquad \cdots|x_1\rangle\langle x_1|e^{-iH(t_f-t_i)/N}|x_0\rangle \end{aligned}$$

This is exact for any N, where we've set $x_0 = x_i$ and $x_N = x_f$. Sending $N \to \infty$ we have repeated evolution over infinitesimal time intervals, and making use of our previous result

$$\langle x_f|e^{-iH(t_f-t_i)}|x_i\rangle = \lim_{N\to\infty}\int dx_1\cdots dx_{N-1}\,\left(\frac{m}{2\pi i\epsilon}\right)^{N/2}\exp\left\{i\epsilon\sum_{i=1}^{N}\left[\frac{1}{2}m\left(\frac{x_i-x_{i-1}}{\epsilon}\right)^2 - V(x_{i-1})\right]\right\}$$

where $\epsilon = (t_f - t_i)/N$. Now we define a measure for integrating over paths

$$\mathcal{D}x(\cdot) = \lim_{N\to\infty}\,dx_1\cdots dx_{N-1}\,\left(\frac{m}{2\pi i\epsilon}\right)^{N/2}$$

and identify the action for a path as

$$S[x(\cdot)] = \lim_{N\to\infty}\,\epsilon\sum_{i=1}^{N}\left[\frac{1}{2}m\left(\frac{x_i-x_{i-1}}{\epsilon}\right)^2 - V(x_{i-1})\right].$$

Then we can write the matrix elements of the time evolution operator as

$$\langle x_f|e^{-iH(t_f-t_i)}|x_i\rangle = \int_{\substack{x(t_i)=x_i\\ x(t_f)=x_f}}\mathcal{D}x(\cdot)\,e^{iS[x(\cdot)]}$$

We have expressed the amplitude for a particle to go from position x_i at time t_i, to position x_f at time t_f, in terms of a sum over all paths connecting (x_i, t_i) to (x_f, t_f). This is known as the Feynman "path integral" or "sum-over-histories". Note that the classical path, *i.e.* the path satisfying the classical equations of motion, does not play a special role here! On the contrary all paths are equally likely, in the sense that the probability of any given path $\sim |\text{amplitude}|^2 \sim |e^{iS[\text{path}]}|^2 \sim 1$. (Question – what is the significance of the classical trajectory?)

This leads to a general attitude towards quantum theory, which we'll take over wholeheartedly into string theory: integrate over everything that isn't fixed by your initial or final conditions.

Your friend, the Gaussian path integral

There's basically only one path integral anyone knows how to evaluate. It's a generalization of the ordinary one-dimensional integral

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}\,.$$

This formula is valid if a is a positive real number; by analytic continuation it also holds in the complex a plane.

As an intermediate step to path integrals let A be an $N \times N$ real symmetric positive-definite matrix, and let's try to make sense of

$$\int_{\mathbb{R}^N} d^N x\, e^{-\mathbf{x}^T A \mathbf{x}}\,.$$

We can diagonalize A with an orthogonal transformation, $A = R^T \Lambda R$ where R is orthogonal and $\Lambda = \text{diag.}(\lambda_1, \ldots, \lambda_N)$ is diagonal with real positive entries. Then setting $\mathbf{y} = R\mathbf{x}$ we have

$$\int d^N x\, e^{-\mathbf{x}^T A \mathbf{x}} = \int d^N y\, e^{-\mathbf{y}^T \Lambda \mathbf{y}} = \frac{\pi^{N/2}}{(\lambda_1 \cdots \lambda_N)^{1/2}} = \frac{\pi^{N/2}}{\det^{1/2} A}\,.$$

Defining our integration measure $\mathcal{D}\mathbf{x} = d^N x / \pi^{N/2}$ we have

$$\int \mathcal{D}\mathbf{x}\, e^{-\mathbf{x}^T A \mathbf{x}} = \det{}^{-1/2} A\,.$$

By analytic continuation we can extend this to complex symmetric A.

Nothing here seems to depend on N, so we might as well send $N \to \infty$ and consider integration over an infinite-dimensional space. In particular let's consider a function space

$$\Big\{\, x(\cdot) \,:\, [0,1] \to \mathbb{R} \,\Big\}$$

with inner product

$$\Big(x(\cdot), y(\cdot)\Big) = \int_0^1 dt\, x(t) y(t)\,.$$

(I'll be sloppy about exactly what space of functions we're considering). Consider a real symmetric positive operator $\mathcal{O}$ with eigenvalues

$$\mathcal{O} x_n(t) = \lambda_n x_n(t) \,.$$

Then with a suitable integration measure we have

$$\int \mathcal{D}x(\cdot)\, e^{-\int_0^1 dt\, x(t)\mathcal{O}x(t)} = \det{}^{-1/2}\mathcal{O} \equiv \prod_{n=1}^{\infty} \lambda_n^{-1/2} \,.$$

We'll often extend this to complex symmetric operators. Two comments,

- A multitude of sins can be swept under the rug by saying "with a suitable integration measure."
- For most operators of interest the determinant (the infinite product) diverges. In QFT you'll learn how to handle this with renormalization.