References: Our discussion of the Polyakov action is quite standard; see for instance Zwiebach section 21.6. Riemann surfaces are treated in Zwiebach section 22.4 and Green, Schwarz and Wittten section 3.3. For a somewhat more mathematical treatment see B. Hatfield, *Quantum field theory of point particles and strings* chapter 21.

Polyakov action

Our starting point is the Polyakov action

$$S[h,X] = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \,. \qquad (1)$$

Here $\sigma^\alpha = (\tau, \sigma)$ are coordinates on the string worldsheet, $h_{\alpha\beta}(\tau,\sigma)$ is an independent dynamical variable (the worldsheet metric), and $X^\mu(\tau,\sigma)$ are scalar fields from the worldsheet point of view that describe the embedding of the worldsheet in a Minkowski target space with metric $\eta_{\mu\nu}$. The string tension T, the Regge slope α', and the string length ℓ_S are related by

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi\ell_S^2} \,.$$

From the two-dimensional point of view this is a rather curious theory: it's two-dimensional gravity (a fluctuating metric) interacting with a collection of scalar fields X^μ. Here we want to show that it's equivalent to the Nambu-Goto action. To do this we vary (1) with respect to the worldsheet metric to find the equation of motion

$$\frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \,. \qquad (2)$$

This requires that the worldsheet metric $h_{\alpha\beta}$ be proportional to the induced metric $\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$, so let's set

$$h_{\alpha\beta} = e^{2\phi} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \qquad (3)$$

and see if we can determine ϕ. Plugging this ansatz back into the equation of motion we find

$$\frac{1}{2} e^{2\phi} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\, h^{\gamma\delta} \underbrace{\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu}}_{=e^{-2\phi} h_{\gamma\delta}} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \,.$$

But this is a trivial identity, satisfied for any ϕ. So the equation of motion for $h_{\alpha\beta}$ does not fix the worldsheet metric uniquely, a fact which will be important in the sequel. All we can say at this point is that the worldsheet metric must be proportional to the induced metric. This is good enough, because if we plug (3) back into the Polyakov action we find

$$\begin{aligned}
&\det h_{\alpha\beta} = e^{4\phi} \det \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \\
&h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} = 2e^{-2\phi} \\
&\Rightarrow \quad S[X] = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}}
\end{aligned}$$

just the usual Nambu-Goto action (the area of the string worldsheet, measured using the induced metric).

There is an important point here. The fact that ϕ dropped out of the Polyakov action reflects an underlying symmetry: the Polyakov action is invariant under arbitrary (position-dependent) Weyl rescalings of the worldsheet metric which act according to

$$h_{\alpha\beta}(\sigma) \to e^{2\omega(\sigma)} h_{\alpha\beta}(\sigma) \qquad X^\mu(\sigma) \text{ unchanged} \tag{4}$$

Wick rotation

At this point we do something that's a little fishy. So far we've had in mind that the metric $h_{\alpha\beta}$ has a Lorentzian signature; locally one could choose coordinates to set $ds^2 = -d\tau^2 + d\sigma^2$. But from now on we're going to "Wick rotate" and take the metric to have a Euclidean signature. Locally we can do this by setting $\tau = -i\tau_E$ so that $ds^2 = d\tau_E^2 + d\sigma^2$. A few comments:

- You might object on the grounds that the equation of motion (2) can no longer be satisfied since the induced metric is Lorentzian (at least for timelike worldsheets). I've never known what to say about that.
- I sort of think that if you work carefully with a Lorentzian signature on the worldsheet you'll be led to the light-cone formalism of GSW chapter 11 (or Zwiebach section 22.3) where the string worldsheets aren't quite manifolds (they have sharp corners). Perhaps the path integral over such Lorentzian not-quite-manifolds agrees with the path integral over smooth Euclidean manifolds?

- The best justification for the whole procedure is probably that, as we'll see, it produces well-behaved string scattering amplitudes.

A few words from the masters:

> We cannot really prove that world-sheets can be treated as Riemann surfaces, but nothing stops us from trying to treat them as such. *Zwiebach p. 490*

> We will take the Euclidean theory as our starting point, however let us give a brief formal argument that it is equivalent to the Minkowski theory with which we began. *Polchinski p. 82*

> It is convenient, though not essential, to use Euclidean language. *GSW p. 124*

String path integral and gauge fixing

Now let's consider the string path integral[1]

$$\int \frac{\mathcal{D}X^\mu(\cdot)\mathcal{D}h_{\alpha\beta}(\cdot)}{\mathsf{Diff}\times\mathsf{Weyl}}\,\exp\left\{-\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}\,h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\right\}$$

Following lessons learned from the point particle, we've taken care to divide by the action of the group Diff of two-dimensional diffeomorphisms, as well as by the group Weyl of local Weyl rescalings. That is, we only want to integrate over equivalence classes of metrics modulo Diff $\times$ Weyl transformations.

For the moment we have in mind a worldsheet Σ of fixed but unspecified topology; we'll say more about the issue of worldsheet topology later on. Let's work locally on Σ, in a chart with coordinates σ^α, and study the question of fixing the symmetry locally. We start with a general worldsheet metric $ds^2 = h_{\alpha\beta}d\sigma^\alpha d\sigma^\beta$. Under a Weyl transformation

$$\hat{h}_{\alpha\beta} = e^{2\omega}h_{\alpha\beta}$$

[1] We're Euclidean now. Setting $d^2\sigma = d\tau d\sigma = -id\tau_E d\sigma$ brought out a $-i$ that cancelled the i in e^{iS}.

the curvature scalar transforms as (Polchinski, p. 85)

$$\sqrt{\hat{h}}\hat{R} = \sqrt{h}\left(R - 2\nabla^2\omega\right) .$$

So we can set $\hat{R} = 0$ by solving $\nabla^2\omega = R/2$. Locally this equation can always be solved (if you like, use your Green's function from the homework). In two dimensions vanishing scalar curvature implies that a space really is flat. Then we can choose coordinates in which $\hat{h}_{\alpha\beta} = \delta_{\alpha\beta}$.

To summarize, it looks like we can fix the $\mathsf{Diff} \times \mathsf{Weyl}$ symmetry by setting $h_{\alpha\beta} = \delta_{\alpha\beta}$. This fits with a naive counting argument: the worldsheet metric has three independent components, and our symmetry transformations depend on three arbitrary functions, so we ought to be able to set $h_{\alpha\beta}$ to whatever we like. In particular we ought to be able to set $h_{\alpha\beta} = \delta_{\alpha\beta}$.

If this conclusion were completely correct, we could cancel the integral over worldsheet metrics against the action of the symmetry group, and write the string path integral as

$$\int \mathcal{D}X^\mu(\cdot)\, \exp\left\{-\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\, \delta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right\}$$

However there are two important caveats to this conclusion.

1. It could be there are geometric quantities ("moduli") characterizing the worldsheet which are invariant under $\mathsf{Diff} \times \mathsf{Weyl}$. If so, rather than completely cancelling the integral over worldsheet metrics, we'd be left with an integral over moduli.

2. It could be that setting $h_{\alpha\beta} = \delta_{\alpha\beta}$ does not completely fix our choice of gauge. If so, there would still be a residual symmetry group that we need to divide by.

Keeping these possibilities in mind, we will write the string path integral as

$$\int \frac{\mathcal{D}(\text{moduli})\mathcal{D}X^\mu(\cdot)}{\text{residual symmetries}}\, \exp\left\{-\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\, \sqrt{\hat{\delta}}\, \hat{\delta}^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right\}$$

where $\hat{\delta}_{\alpha\beta}$ is a little subtle: it doesn't really mean the trivial flat metric, rather it denotes a representative element of the equivalence class of metrics specified by the values of the moduli. So $\hat{\delta}_{\alpha\beta}$ might be trivially flat in any given coordinate chart but have some non-trivial global structure.

Conformal invariance

You can see there's a potential residual symmetry group just working locally on the worldsheet. Consider a chart in which we've fixed a trivial flat metric $ds^2 = d\tau^2 + d\sigma^2$. It's convenient to introduce complex coordinates

$$z = \tau + i\sigma \qquad \bar{z} = \tau - i\sigma$$

in which $ds^2 = dzd\bar{z}$. That is, we've fixed the gauge by setting

$$h_{zz} = h_{\bar{z}\bar{z}} = 0 \qquad h_{z\bar{z}} = h_{\bar{z}z} = 1/2\,. \tag{5}$$

Now consider a general change of coordinates

$$z \to z' = f(z,\bar{z}) \qquad \bar{z} \to \bar{z}' = \bar{f}(z,\bar{z})$$

under which

$$\begin{aligned}
dz &\to \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \\
d\bar{z} &\to \frac{\partial \bar{f}}{\partial z}dz + \frac{\partial \bar{f}}{\partial \bar{z}}d\bar{z} \\
ds^2 &\to \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial z}dz^2 + \left(\left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}\right|^2\right)dzd\bar{z} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}d\bar{z}^2
\end{aligned}$$

In the new coordinate system we no longer have (5). However suppose the change of coordinates is analytic, $\frac{\partial f}{\partial \bar{z}} = \frac{\partial \bar{f}}{\partial z} = 0$. Then

$$ds^2 \to \left|\frac{\partial f}{\partial z}\right|^2 dzd\bar{z}$$

and making a Weyl transformation $ds^2 \to \frac{1}{|f'|^2}ds^2$ takes us back to $ds^2 = dzd\bar{z}$.

To summarize, our local choice of gauge $ds^2 = dzd\bar{z}$ is preserved by the combined $\mathsf{Diff} \times \mathsf{Weyl}$ transformation

$$z \to f(z) \qquad \bar{z} \to \bar{f}(\bar{z}) \qquad ds^2 \to \frac{1}{|f'|^2}ds^2\,.$$

This defines a subgroup $\mathsf{Conf} \subset \mathsf{Diff} \times \mathsf{Weyl}$ known as the local conformal group. Two comments,

1. Although we've identified a large potential residual symmetry group, analytic changes of coordinate are measure zero in the space of all diffeomorphisms, so it's not surprising our previous parameter-counting arguments missed this subtlety.

2. Our gauge choice is also preserved by $z \to z^*$, however we won't have much use for such orientation-reversing transformations.

Riemann surfaces

At this point we have to face up to global topological issues. We consider compact oriented two-dimensional manifolds Σ without boundaries. Such manifolds are classified topologically by their genus g, which just counts the number of handles: the sphere is genus zero, the torus is genus 1, etc. A remarkable theorem relates the Euler characteristic χ to the integral of the scalar curvature,

$$\chi(\Sigma) = 2 - 2g = \frac{1}{4\pi} \int d^2\sigma \, \sqrt{h} R \, .$$

This shows there's an obstruction to setting $h_{\alpha\beta} = \delta_{\alpha\beta}$ everywhere on a Riemann surface (unless you happen to be working on a torus). So our gauge choice does bump up against global topology. There are two main questions we want to address.

- What is the moduli space, i.e. the space $\{\text{metrics}\}/\mathsf{Diff} \times \mathsf{Weyl}$?

- What is the group of globally-defined conformal transformations?

The answers to these questions depend in a rather intricate way on the topology of the worldsheet. We'll study this in a pedestrian case-by-case way.

<u>The sphere</u>

Let's start with the sphere. Given any space with genus zero we can make a Weyl transform to turn it into a smooth round two-sphere of arbitrary radius r, which we can represent as

$$x^2 + y^2 + z^2 = r^2 \quad \text{embedded in } \mathbb{R}^3 \, .$$

Defining coordinates on S^2 by projecting from the north pole to the xy plane, after a somewhat tedious calculation we can write the metric as

$$ds^2 = \frac{4r^4(dx^2 + dy^2)}{(x^2 + y^2 + r^2)^2} .$$

It's convenient to introduce complex coordinates

$$z = x + iy \qquad \bar{z} = x - iy$$

and write the metric of the Riemann sphere as

$$ds^2 = \frac{4r^4}{\left(|z|^2 + r^2\right)^2} dz d\bar{z} . \tag{6}$$

By an additional Weyl transformation we can drop the conformal factor and take the metric to be $ds^2 = dzd\bar{z}$. But this is a bit of a cheat – you should keep in mind that there really is some curvature hidden at infinity. What this construction does show is that the moduli space of $\{\text{metrics}\}/\mathsf{Diff} \times \mathsf{Weyl}$ is trivial on S^2 – any two metrics can be mapped into each other by a $\mathsf{Diff} \times \mathsf{Weyl}$ transformation. So there are no moduli to integrate over.

What about the group of globally-defined conformal transformations? Based on our local discussion, any analytic change of coordinates is a good candidate. Specializing to infinitesimal transformations, the candidate conformal transformations are given by holomorphic vector fields

$$z \longrightarrow z + \delta z \qquad \delta z = \epsilon^z = \sum_{n=0}^{\infty} a_n z^n .$$

We restrict to $n \geq 0$ so the vector field is non-singular at $z = 0$. However that's not the only condition we need to impose. We also need to worry about behavior near infinity (which just corresponds to the north pole of the sphere, so nothing singular should happen there). The easiest way to study this is to set $w = 1/z$. This map takes $z = \infty$ to $w = 0$, and in the new coordinate our vector field becomes

$$\epsilon^w = \frac{\partial w}{\partial z} \epsilon^z = -\frac{1}{z^2} \epsilon^z = -\sum_{n=0}^{\infty} a_n w^{2-n} .$$

Only for $n = 0, 1, 2$ do we have a vector field that is non-singular everywhere on the Riemann sphere. So we have a 3-complex-parameter space of globally-defined infinitesimal conformal transformations.

$$\delta z = \alpha + \beta z + \gamma z^2$$
$$\delta \bar{z} = \bar{\alpha} + \bar{\beta}\bar{z} + \bar{\gamma}\bar{z}^2$$

These are known as conformal Killing vectors. The finite form of these global conformal transformations is

$$z \to \frac{az+b}{cz+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})\,.$$

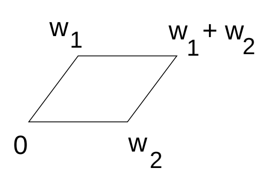
Here $SL(2,\mathbb{C})$ is the group of 2×2 complex matrices with unit determinant. See Polchinski, p. 168 or GSW p. 384.

Bottom line: the sphere has no moduli, while the global conformal group is $SL(2,\mathbb{C})$, so the string path integral is

$$\int \frac{\mathcal{D}X^\mu(\cdot)}{SL(2,\mathbb{C})} \, \exp\left\{-\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\delta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right\}$$

<u>The torus</u>

Next let's consider the torus. Starting from any genus-1 manifold we can make the curvature vanish everywhere with a Weyl transformation. The resulting flat torus can be obtained by identifying opposite sides of a parallelogram, or equivalently by periodically identifying the complex plane.



$$z \approx z + w_1 \approx z + w_2 \qquad w_1, w_2 \in \mathbb{C}\,.$$

By a diffeomorphism (a rotation of the complex plane) we can make w_2 real, and by a constant Weyl rescaling we can set $w_2 = 1$. However we are left with one Diff $\times$ Weyl-invariant quantity which characterizes our torus: the modular parameter τ, in general given by $\tau = w_1/w_2$ (sorry, τ is standard notation, it has nothing to do with worldsheet time τ). As another way to see that τ characterizes conformally-inequivalent tori, note that the two fundamental cycles on the torus have length $|w_1|$ and $|w_2|$, respectively, and that the angle between the fundamental cycles is $\arg(w_1/w_2)$. Both the ratio of lengths $|\tau|$ and the angle $\arg\tau$ are conformally-invariant quantities.

So the torus has a single complex modulus. What about the global conformal group? As on the Riemann sphere, the candidate infinitesimal conformal transformations are holomorphic vector fields

$$z \rightarrow z + \delta z \qquad \delta z = \epsilon^z = \sum_n a_n z^n .$$

However only for $n = 0$ does the vector field have the right periodicity to be single-valued on the torus. So the global conformal group of the torus is just the group of rigid translations

$$z \rightarrow z + a \qquad \bar{z} \rightarrow \bar{z} + \bar{a} .$$

For future reference, it will be useful to say a little more about the moduli space of genus-1 Riemann surfaces – that is, to address the question: what are the allowed values of τ? This is discussed in Polchinski, pp. 146–149 and in GSW vol. I, pp. 157–159 and vol. II pp. 44–50. There's also an extensive discussion in Hatfield, pp. 579–591. First, to keep our torus from degenerating, we restrict to $\operatorname{Im}\tau > 0$. Thus the "Teichmuller space," the space of metrics modulo $\mathsf{Diff}_0 \times \mathsf{Weyl}$, is the upper half plane $\mathbb{H}$. Here Diff_0 denotes the group of diffeomorphisms which are continuously connected to the identity.

But Diff_0 isn't the end of the story. Consider the lattice of points in the complex plane

$$\Gamma = \{n_1 w_1 + n_2 w_2 | n_1, n_2 \in \mathbb{Z}\}$$

where the basis vectors are $w_1 = \tau$, $w_2 = 1$. If we make a $GL(2,\mathbb{Z})$ transformation on the basis vectors,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} ,$$

we get exactly the same lattice: one just has to make a compensating $GL(2,\mathbb{Z})$ transformation on the integers (n_1, n_2) to get the same point in the complex plane. ($GL(2,\mathbb{Z})$ is the group of 2×2 matrices with integer entries and determinant ± 1.)

We can think of our torus as $\mathbb{C}/\Gamma$, i.e. we identify points in $\mathbb{C}$ that differ by a lattice vector. Note that $GL(2,\mathbb{Z})$ acts on the modular parameter $\tau = w_1/w_2$ according to

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} .$$

In particular it acts on the imaginary part as ($\tau = \tau_1 + i\tau_2$)

$$\tau_2 \to (ad - bc)\frac{\tau_2}{|c\tau + d|^2}.$$

In order that the transformation preserve $\operatorname{Im}\tau > 0$ we must have $ad - bc = +1$, i.e. we must restrict to the group $SL(2,\mathbb{Z})$. Actually, both $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ have the same effect on the modular parameter. So the group which acts on τ is $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ subgroup is $\{\mathbb{1}, -\mathbb{1}\}$. This is known as the modular group of the torus. It can be thought of as the group of large diffeomorphisms of the torus (diffeomorphisms which are not continuously connected to the identity).

So the true moduli space of $\{\text{metrics}\}/\mathsf{Diff} \times \mathsf{Weyl}$ is given by $\mathbb{H}/PSL(2,\mathbb{Z})$. To identify this space we note that $PSL(2,\mathbb{Z})$ is generated by the elements

$$\begin{aligned} S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad & S : \tau \to -1/\tau \\ T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad & T : \tau \to \tau + 1 \end{aligned}$$

By making $PSL(2,\mathbb{Z})$ transformations we can restrict τ to lie in the 'fundamental modular domain'

$$\mathcal{F} = \left\{ \tau \in \mathbb{C} \,:\, \operatorname{Im}\tau > 0, \quad |\tau| > 1, \quad -\frac{1}{2} < \operatorname{Re}\tau < \frac{1}{2} \right\}$$

See GSW, vol. II, pp. 49 or Polchinski pp. 148–149.

Bottom line: the torus has one complex modulus $\tau \in \mathcal{F}$, and the global conformal group is just rigid translations $z \to z + \text{const.}$, so the string path integral is

$$\int \frac{\mathcal{D}\tau \mathcal{D}X^\mu(\cdot)}{\text{translations}} \exp\left\{ -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma\, \delta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \right\}$$

You might ask, what's the right measure for integrating over τ? I claim it's

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\, \det{}'(-\nabla^2)$$

There are two ingredients here.

1. The claim that the measure for integrating over constant metrics on the torus is $d^2\tau/\tau_2$ where $d^2\tau = d(\mathrm{Re}\,\tau)d(\mathrm{Im}\,\tau)$.

2. The claim that the change of variables from fluctuating metrics to non-constant diffeomorphisms gives rise to a determinant $\det{}'(-\nabla^2)$, where $\det{}'$ means product of non-zero eigenvalues.

You can establish these properties with the same logic you used for the point particle on the last homework.

One can show that $\det{}'(-\nabla^2) = \tau_2^2|\eta(\tau)|^4$ where the Dedekind eta-function

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q = e^{i2\pi\tau}\,.$$

Also with a suitable covariant norm the volume of the group of translations is just

$$\mathrm{vol}(\text{translation group}) = \tau_2^2\,.$$

Putting this all together, the path integral on the torus is

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\,\tau_2|\eta(\tau)|^4 \int \mathcal{D}X^\mu(\cdot)\,\exp\left\{-\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\delta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\right\}$$

I've written the measure for integrating over τ in a slightly funny way to make it clear that it's invariant under the modular or $PSL(2,\mathbb{Z})$ transformations $\tau \to \tau+1$ and $\tau \to -1/\tau$. It's easy to see that $d^2\tau/\tau_2^2$ is modular-invariant; to show that $\tau_2|\eta(\tau)|^4$ is also modular-invariant one needs the results (Polchinski p. 214–216)

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$
$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)\,.$$

This makes it consistent to restrict the τ integral to the fundamental domain $\mathcal{F}$.

Higher genus

What happens at higher genus? One can show that there are no conformal Killing vectors for genus $g \geq 2$ (Polchinski, p. 152), so there are no residual symmetries to worry about. On the other hand an index theorem (Polchinski p. 158) shows that a Riemann surface with genus $g \geq 2$ can be characterized by $6g-6$ real moduli.