

References: A good reference for this material is Green, Schwarz and Witten, *Superstring theory*, volume I. The Virasoro amplitude is covered in sections 1.4.2 – 1.4.4 and revisited in section 7.2; see p. 385 for a discussion of $SL(2, \mathbb{C})$ gauge fixing. Strings in background fields are discussed in sections 3.4.5 and 3.4.6.

String scattering

Now let's see if we can use all this formalism to study closed string scattering. Our starting point will be the path integral for a string worldsheet Σ with the topology of a sphere,

$$\int \frac{\mathcal{D}X^\mu(\cdot)}{SL(2, \mathbb{C})} \exp \left\{ -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \delta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \right\} \quad (1)$$

We'll discuss the significance of a spherical worldsheet topology later on.

Our first problem is that, as written, the path integral (1) doesn't seem to have anything to do with scattering. Let's draw some inspiration from point particles, where we first worked out the amplitude for the particle to propagate between fixed points x_i^μ and x_f^μ in spacetime, then switched to momentum space to discuss scattering.

To carry out the analogous procedure in string theory is pretty straightforward. Let's fix N points in spacetime x_1^μ, \dots, x_N^μ and require that our string worldsheet pass through these N points. We can do this just by inserting a bunch of δ -functions in our path integral.

$$\begin{aligned} \mathcal{M}(x_1, \dots, x_N) &= \int \frac{\mathcal{D}X^\mu(\cdot)}{SL(2, \mathbb{C})} \int d^2\sigma_1 \delta^D(X(\sigma_1) - x_1) \cdots \int d^2\sigma_N \delta^D(X(\sigma_N) - x_N) \\ &\quad \exp \left\{ -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \delta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \right\} \end{aligned} \quad (2)$$

This is known as a pinned path integral: Polyakov, p. 192. We integrate over the positions $\sigma_1, \dots, \sigma_N$ because we don't care which points on the worldsheet pass through our given locations in spacetime.

If we want to describe a scattering process, we aren't interested in fixing initial and final positions. Rather we want to fix incoming and outgoing momenta. This is easy

to do: we just Fourier transform and act on (2) with $\int d^D x_1 e^{ik_1 \cdot x_1} \dots \int d^D x_N e^{ik_N \cdot x_N}$.¹ This leads to

$$\begin{aligned} \mathcal{M}(k_1, \dots, k_N) = & \int \frac{\mathcal{D}X^\mu(\cdot)}{SL(2, \mathbb{C})} \int d^2 \sigma_1 e^{ik_1 \cdot X(\sigma_1)} \dots \int d^2 \sigma_N e^{ik_N \cdot X(\sigma_N)} \\ & \exp \left\{ -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 \sigma \delta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \right\} \end{aligned} \quad (3)$$

We identify this as the amplitude for scattering N closed strings. One point we might revisit: since we haven't excited any quantum numbers on the incoming or outgoing strings, this is the amplitude for scattering of strings in their ground state. With bosonic closed strings this means we're scattering tachyons with $m^2 = -4/\alpha'$. Also, some terminology: quantities like $e^{ik \cdot X(\sigma)}$ are known as “vertex operators.”

$SL(2, \mathbb{C})$ fixing

We seem to be in good shape because the X^μ path integral in (3) is Gaussian. But there's one sticky issue we need to address before doing the path integral: how do we divide by the global conformal group $SL(2, \mathbb{C})$?

It's convenient to use complex coordinates and write the amplitude as

$$\mathcal{M}(k_1, \dots, k_N) = \int \frac{\mathcal{D}X^\mu(\cdot)}{SL(2, \mathbb{C})} \int d^2 z_1 e^{ik_1 \cdot X(z_1)} \dots \int d^2 z_N e^{ik_N \cdot X(z_N)} e^{-S[X]} \quad (4)$$

Recall that the global conformal group acts on the Riemann sphere as

$$z \rightarrow \frac{az + b}{cz + d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

We have a three (complex) parameter group of transformations which we can use to set

$$z_1 = z_1^0 \quad z_2 = z_2^0 \quad z_3 = z_3^0$$

where z_1^0, z_2^0, z_3^0 are three arbitrary but fixed positions in the complex plane. More formally we're going to write z_1, z_2, z_3 as an $SL(2, \mathbb{C})$ transformation acting on some fixed positions z_1^0, z_2^0, z_3^0 and we're going to change variables of integration from

¹I've oriented all momenta so they're flowing into the diagram. By momentum conservation we'd better have $\sum_i k_i = 0$. If you want an outgoing momentum just flip the sign, $k \rightarrow -k$.

$\int d^2 z_1 d^2 z_2 d^2 z_3$ to an integral over $SL(2, \mathbb{C})$. When we do this we pick up a Jacobian (it's $|\det|^2$ because $d^2 z = dz d\bar{z}$).

$$\int d^2 z_1 d^2 z_2 d^2 z_3 = \int \mathcal{D}(SL(2, \mathbb{C})) \left| \det \frac{\delta z_i}{\delta SL(2, \mathbb{C})} \right|_{z_i=z_i^0}^2$$

To compute the determinant we use the infinitesimal form of the $SL(2, \mathbb{C})$ transformation

$$\delta z = \alpha + \beta z + \gamma z^2 \quad (5)$$

which implies

$$\begin{aligned} dz_1 dz_2 dz_3 &= d\alpha d\beta d\gamma \det \begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{pmatrix} \\ &= d\alpha d\beta d\gamma (z_1 - z_2)(z_2 - z_3)(z_3 - z_1) \end{aligned}$$

that is²

$$\int d^2 z_1 d^2 z_2 d^2 z_3 = \int \mathcal{D}(SL(2, \mathbb{C})) |z_1^0 - z_2^0|^2 |z_2^0 - z_3^0|^2 |z_3^0 - z_1^0|^2$$

Using this in our scattering amplitude (4) the integral over $SL(2, \mathbb{C})$ cancels and we're left with

$$\begin{aligned} \mathcal{M}(k_1, \dots, k_N) &= \int d^2 z_4 \cdots d^2 z_N |z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2 \\ &\int \mathcal{D}X^\mu(\cdot) e^{ik_1 \cdot X(z_1)} \cdots e^{ik_N \cdot X(z_N)} \exp \left\{ -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \delta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \right\} \end{aligned} \quad (6)$$

where I've dropped the superscripts 0 on the fixed positions z_1, z_2, z_3 .

²I'm making an implicit assumption here, that $d\alpha d\beta d\gamma$ is the correct invariant (Haar) measure for integrating over $SL(2, \mathbb{C})$. To see this note that the infinitesimal transformation (5) corresponds to a matrix $\mathbf{1} + T \in SL(2, \mathbb{C})$ where $T = \begin{pmatrix} \beta/2 & \alpha \\ -\gamma & -\beta/2 \end{pmatrix}$ is an element of the Lie algebra. T has norm $\text{Tr} T^2 = \frac{1}{2}\beta^2 - 2\alpha\gamma$, i.e. the metric on $SL(2, \mathbb{C})$ is $ds^2 = \frac{1}{2}d\beta^2 - 2d\alpha d\gamma$ with volume element $\sim d\alpha d\beta d\gamma$.

The Virasoro amplitude

Now let's evaluate the path integral in (6). As a warm up, note that the one-dimensional integral

$$\int_{-\infty}^{\infty} dx e^{ikx} e^{-ax^2/4\pi\alpha'} = \left(\frac{4\pi^2\alpha'}{a}\right)^{1/2} e^{-\pi\alpha'k^2/a}$$

has an N -dimensional generalization

$$\int d^N x e^{i\mathbf{k}\cdot\mathbf{x}} \exp\left\{-\frac{1}{4\pi\alpha'}\mathbf{x}^T A \mathbf{x}\right\} \sim \frac{1}{\sqrt{\det A}} \exp\{-\pi\alpha'\mathbf{k}^T A^{-1}\mathbf{k}\}.$$

To extend this to our path integral we identify $A = -\nabla^2$ with minus the Laplacian, $A^{-1} = -\frac{1}{2\pi} \log \mu|z|$ with minus the Green's function you constructed on the homework, and $\mathbf{k} = \sum_i k_i \delta^2(z - z_i)$ with the incoming momenta. Then our path integral is

$$\begin{aligned} & \int \mathcal{D}X^\mu(\cdot) e^{ik_1 \cdot X(z_1)} \dots e^{ik_N \cdot X(z_N)} \exp\left\{-\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \delta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right\} \\ &= \det^{-1/2}(-\nabla^2) \exp\left\{\frac{1}{2}\alpha' \sum_{i,j} k_i \cdot k_j \log \mu|z_i - z_j|\right\} \\ &= \det^{-1/2}(-\nabla^2) \prod_{i,j} (\mu|z_i - z_j|)^{\alpha' k_i \cdot k_j / 2} \end{aligned}$$

A few comments:

1. We'll drop the $\det^{-1/2}(-\nabla^2)$ since it's just an overall factor in the amplitude (it doesn't depend on the external momenta).
2. Likewise since $k_i^2 = -m^2 = 4/\alpha'$ terms in the product with $i = j$ just contribute a (divergent) overall factor and can be neglected.
3. Finally the amplitude is independent of μ since $\sum_i k_i^\mu = 0$ implies $\sum_{i,j} k_i \cdot k_j = 0$.

Plugging these results back into our amplitude we have

$$\mathcal{M}(k_1, \dots, k_N) = \int d^2 z_4 \dots d^2 z_N |z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2 \prod_{i < j} |z_i - z_j|^{\alpha' k_i \cdot k_j}$$

For simplicity let's specialize to a four-tachyon scattering amplitude. Also let's set $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$.³ Then we're left with the Virasoro amplitude

$$\mathcal{M}(k_1, k_2, k_3, k_4) = \int d^2 z_4 |z_4|^{\alpha' k_1 \cdot k_4} |1 - z_4|^{\alpha' k_2 \cdot k_4}. \quad (7)$$

For most purposes it's best to leave the Virasoro amplitude in the form (7). However one can evaluate the z integral in terms of Euler Γ -functions to find

$$\mathcal{M}(s, t, u) = \frac{\pi \Gamma(-1 - \alpha' s/4) \Gamma(-1 - \alpha' t/4) \Gamma(-1 - \alpha' u/4)}{\Gamma(2 + \alpha' s/4) \Gamma(2 + \alpha' t/4) \Gamma(2 + \alpha' u/4)} \quad (8)$$

The necessary manipulations are described in GSW p. 386. Here we've introduced some standard notation, the Mandelstam invariants

$$\begin{aligned} s &= -(k_1 + k_2)^2 = 4E^2 \\ t &= -(k_1 + k_3)^2 = -4p^2 \sin^2(\theta/2) \\ u &= -(k_1 + k_4)^2 = -4p^2 \cos^2(\theta/2) \end{aligned}$$

where E , p and θ are the energy, momentum and scattering angle in the center of mass frame. These invariants completely characterize the kinematics. Note that they are not independent, but rather obey $s + t + u = 4m^2$.

In either of the forms (7) or (8) it's not obvious that the Virasoro amplitude is worthy of much attention. But scattering amplitudes of this type, first written down by Veneziano, were really the beginnings of string theory. In particular, as you'll show on the homework, *they're compatible with unitarity even though string theory has a graviton in the spectrum!*

Strings in background fields and the dilaton

I'd like to return to the issue of worldsheet topology: what's the significance of having a spherical worldsheet versus some more complicated topology? It turns out the best way to get a handle (sorry, bad pun) on this is to study strings in background fields, a subject which is of interest in its own right.

³The worrisome z_3 dependence drops out if you use momentum conservation $\sum_i k_i^\mu = 0$ and the on-shell conditions $k_i^2 = 4/\alpha'$.

Recall the Polyakov action for a string in Minkowski space,

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. It seems pretty obvious that if we wanted to describe a string in a curved spacetime we'd use the action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X)$$

where $g_{\mu\nu}$ is the spacetime metric. But besides the graviton (which corresponds to $g_{\mu\nu}$) the closed string spectrum also includes an antisymmetric Kalb-Ramond field $b_{\mu\nu}$, a dilaton ϕ and a tachyon T , as well as infinitely many massive string modes. Can we describe a string moving in a non-trivial background for all these fields? The massive modes are problematic, since so much energy is required to excite them, but for the others we can use a generalized Polyakov action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \left\{ h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) \right. \\ \left. + i\epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu b_{\mu\nu}(X) \right. \\ \left. + \alpha' R(h)\phi(X) + 4\pi\alpha' T(X) \right\}$$

This action is known as a non-linear sigma model for obscure historical reasons. Notation: $\epsilon^{\alpha\beta}$ is an antisymmetric tensor (a volume form) on the worldsheet, with the factor of i arising from Wick rotation, and $R(h)$ is the curvature of the worldsheet metric. We put factors of α' in the action to keep the background fields dimensionless.

How does this help with understanding worldsheet topology? For simplicity suppose the dilaton is constant, $\phi(x) = \phi_0$. In the spirit of path integrals we should sum over all possible topologies of the worldsheet. For closed strings this just means a sum over the genus g . That is, the string ‘‘path integral’’ should really be

$$\sum_{g=0}^{\infty} \int \frac{\mathcal{D}X^\mu(\cdot) \mathcal{D}h_{\alpha\beta}(\cdot)}{\text{Diff} \times \text{Weyl}} \Big|_{\text{genus } g} e^{-S}$$

Now recall that the Euler characteristic $\chi = 2 - 2g = \frac{1}{4\pi} \int d^2\sigma \sqrt{h} R(h)$. So for constant dilaton we can bring out the dilaton dependence explicitly and write the string path integral as

$$\sum_{g=0}^{\infty} e^{(2g-2)\phi_0} \int \frac{\mathcal{D}X^\mu(\cdot) \mathcal{D}h_{\alpha\beta}(\cdot)}{\text{Diff} \times \text{Weyl}} \Big|_{\text{genus } g} e^{-S}$$

with no dilaton left in the action. This means the sum over genus is an expansion in powers of the string coupling $g_s = e^{\phi_0}$. For $g_s \ll 1$ (meaning $\phi_0 \rightarrow -\infty$) the string is weakly coupled and spherical worldsheets dominate. When $g_s \gtrsim 1$ the string is strongly coupled and all worldsheet topologies must be taken into account.