References: Zwiebach chapter 17 and Polchinski chapter 8. There's also nice stuff in GSW section 6.4.

Compactification on S^1

Let's start by studying the simplest possible string compactification: the closed oriented bosonic string on $\mathbb{R}^{1,25} \times S^1$. But first, let's recall the "old covariant" quantization procedure in uncompactified Minkowski space. We begin with the mode expansion

$$X^\mu(\tau,\sigma) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)}\right)$$

The Virasoro generators are

$$\begin{aligned} L_m &= \frac{1}{2}\sum_n : \alpha_n \cdot \alpha_{m-n} : \qquad (1) \\ \tilde{L}_m &= \frac{1}{2}\sum_n : \tilde{\alpha}_n \cdot \tilde{\alpha}_{m-n} : \end{aligned}$$

where the colons denote normal ordering (annihilation operators to the right), and we've made the convenient definitions

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$$

Physical states of the string should satisfy

$$\begin{aligned} (L_0 - 1)|\text{phys}\rangle &= (\tilde{L}_0 - 1)|\text{phys}\rangle = 0 \\ L_m|\text{phys}\rangle &= \tilde{L}_m|\text{phys}\rangle = 0 \qquad \forall m > 0 \end{aligned}$$

We also need to identify two physical states that differ by addition of a null state (a physical state of the form $L_{n<0}|\text{something}\rangle$).

How does this story change if we periodically identify one of the coordinates in the target space, say $x^{25} \approx x^{25} + 2\pi R$? The mode expansions for the uncompactified coordinates X^μ stay the same. But the mode expansion for X^{25} gets modified to

$$X^{25}(\tau,\sigma) = x + \alpha' p\tau + wR\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n e^{-in(\tau+\sigma)} + \tilde{\alpha}_n e^{-in(\tau-\sigma)}\right) \qquad (2)$$

Notation: From now on I'll use indices $\mu, \nu = 0, \ldots, 24$ to denote the uncompactified coordinates. Also I won't bother putting a superscript 25 on quantities like x or p that appear in the mode expansion of X^{25}.

The new feature is the term $wR\sigma$ that appears in the mode expansion of X^{25}. Here w is an integer, the "winding number" of the string. It controls the boundary conditions on the string, in the sense that

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau,\sigma) + 2\pi R w\,.$$

If you like, the worldsheet coordinates X^μ define a map from the worldsheet S^1 to the target space S^1, and w fixes the homotopy class of this map.

Another new feature, not apparent from the notation, is that the momentum in the x^{25} direction must be quantized. This is nothing but the familiar quantization condition for a particle moving on a circle in quantum mechanics. Take a particle in a momentum eigenstate, with a wavefunction $\psi(x) = e^{ipx}$. When x is periodically identified to make a circle, we must have $p = n/R$ with $n \in \mathbb{Z}$ so that the wavefunction is single-valued. The same holds true in string theory: in (2) we must have $p = n/R$.

It's convenient to make a left / right split of the worldsheet coordinates, and set

$$\begin{aligned}
X^{25} &= X_L(\tau+\sigma) + X_R(\tau-\sigma) \\
X_L &= x_L + \frac{1}{2}\alpha' p_L(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha^\mu_n e^{-in(\tau+\sigma)} \\
X_R &= x_R + \frac{1}{2}\alpha' p_R(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-in(\tau-\sigma)}
\end{aligned}$$

Clearly we need $x_L + x_R = x$. More importantly, the left- and right-moving momenta are

$$\begin{aligned}
p_L &= \frac{n}{R} + \frac{wR}{\alpha'} \\
p_R &= \frac{n}{R} - \frac{wR}{\alpha'}
\end{aligned}$$

The new feature, relative to the uncompactified theory, is that in general the left- and right-moving momenta are not equal.

Defining $\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\, p_L$, $\tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\, p_R$ the Virasoro generators are still given by (1). The most important constraints come from the Virasoro zero modes. Writing them out explicitly

$$\begin{aligned} L_0 &= \frac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n \\ &= \frac{\alpha'}{4}\left(p_\mu p^\mu + p_L^2\right) + N \\ \tilde{L}_0 &= \frac{1}{2}\tilde{\alpha}_0 \cdot \tilde{\alpha}_0 + \sum_{n>0} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \\ &= \frac{\alpha'}{4}\left(p_\mu p^\mu + p_R^2\right) + \tilde{N} \end{aligned}$$

In the first lines $\cdot$ denotes the inner product in 26 dimensions. In the second lines we've introduced the left- and right-moving oscillator numbers N, $\tilde{N}$. The mass as seen in the uncompactified directions is $m^2 = -p_\mu p^\mu$ (recall that $\mu = 0, \ldots, 24$). So the constraints

$$(L_0 - 1)|\mathrm{phys}\rangle = (\tilde{\mathrm{L}}_0 - 1)|\mathrm{phys}\rangle = 0$$

imply that

$$\begin{aligned} m^2 &= p_L^2 + \frac{4}{\alpha'}(N-1) \\ m^2 &= p_R^2 + \frac{4}{\alpha'}(\tilde{N}-1) \end{aligned}$$

It's convenient to take the average and difference of these two constraints, to obtain the "mass-shell condition"

$$m^2 = \frac{1}{2}(p_L^2 + p_R^2) + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

and the "level-matching condition"

$$p_L^2 - p_R^2 = \frac{4}{\alpha'}(\tilde{N} - N)\,.$$

These are the most important results for S^1 compactification. In terms of the momentum and winding numbers they read

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{3}$$

$$nw = \tilde{N} - N \tag{4}$$

These expressions have nice physical interpretations. First, the mass-shell condition. It has three contributions:

1. Take a particle (or string) of mass M and give it n units of momentum around a circle. The higher-dimensional mass-shell condition $-p \cdot p = M^2$ becomes $-p_\mu p^\mu - (n/R)^2 = M^2$. So the momentum contributes to the mass as seen in the uncompactified directions: $m^2 = -p_\mu p^\mu = M^2 + (n/R)^2$.

2. Consider a string with w units of winding. The energy of the string is (tension)$\times$ (length), or
$$E = \frac{1}{2\pi\alpha'}\, 2\pi R W = wR/\alpha' \,.$$
As seen in the uncompactified dimensions, the wound string behaves like a particle with $m^2 = (wR/\alpha')^2$.

3. Finally there is an oscillator contribution to the string mass, just as there is in uncompactified space.

The only thing that isn't obvious is that these three contributions should just be added up to give the total (mass)^2 of the string.

To appreciate the level-matching condition, consider a string with $w = 1$. Suppose we want the string to carry n units of momentum as well. The level-matching condition requires that we excite some oscillators, so that $\tilde{N} - N = n$. *The only way to give a wound string momentum around a compact direction is to excite some ripples that travel along the string.*

The string spectrum

Now let's work out the first few states in the string spectrum.

No oscillators

With no oscillators excited the level-matching condition requires $nw = 0$. If $n = w = 0$ we have the usual tachyon, with $m^2 = -4/\alpha'$. Giving the tachyon some momentum around the circle we get an infinite tower of states

$$|n, w = 0\rangle \quad \text{with } m^2 = \tfrac{n^2}{R^2} - \tfrac{4}{\alpha'} \,.$$

The other option is to wind the tachyon around the circle to get

$$|n = 0, w\rangle \quad \text{with } m^2 = \tfrac{w^2 R^2}{\alpha'^2} - \tfrac{4}{\alpha'} \,.$$

One oscillator

If we excite a single right-moving oscillator ($\tilde{N} = 1$, $N = 0$) the level-matching condition requires $nw = 1$. So we get states

$$\tilde{\alpha}^{\mu}_{-1}|n = 1, w = 1\rangle \tag{5}$$
$$\tilde{\alpha}^{\mu}_{-1}|n = -1, w = -1\rangle \tag{6}$$
$$\tilde{\alpha}^{25}_{-1}|n = 1, w = 1\rangle \tag{7}$$
$$\tilde{\alpha}^{25}_{-1}|n = -1, w = -1\rangle \tag{8}$$

All these states have $m^2 = 1/R^2 + R^2/\alpha'^2 - 2/\alpha'$. The first two transform as a vector in 25 dimensions, the last two transform as scalars. Likewise if we excite a single left-moving oscillator we need $nw = -1$, and we get states

$$\alpha^{\mu}_{-1}|n = 1, w = -1\rangle \tag{9}$$
$$\alpha^{\mu}_{-1}|n = -1, w = 1\rangle \tag{10}$$
$$\alpha^{25}_{-1}|n = 1, w = -1\rangle \tag{11}$$
$$\alpha^{25}_{-1}|n = -1, w = 1\rangle \tag{12}$$

Two oscillators

There are quite a few states one can build with two oscillators. Let's concentrate on the massless ones, which have $N = \tilde{N} = 1$ and $n = w = 0$. Then we have states

$$\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|n = w = 0\rangle$$

which can be decomposed into the graviton, dilaton and antisymmetric tensor in 25 dimensions. We also get two gauge fields

$$\alpha^{\mu}_{-1}\tilde{\alpha}^{25}_{-1}|n = w = 0\rangle \tag{13}$$

$$\alpha^{25}_{-1}\tilde{\alpha}^{\mu}_{-1}|n = w = 0\rangle \tag{14}$$

Linear combinations of these fields arise from the $g_{\mu,25}$ and $b_{\mu,25}$ components of the higher-dimensional metric and b-field. The "electric charges" for these gauge fields are the (conserved) momentum and winding around the compactified dimension. Finally we get a scalar field, the radion

$$\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}|n = w = 0\rangle \, .$$

This arises from the $g_{25,25}$ component of the higher-dimensional metric.

Enhanced gauge symmetry

One remarkable feature of the string spectrum: there are extra massless states at special values of the radius. The most interesting case is $R = \sqrt{\alpha'}$. At this radius we pick up four extra massless gauge fields (5), (6), (9), (10), besides the two we always have in (13), (14). We also get a bunch of extra massless scalars.

This phenomenon has an interesting spacetime perspective. For most values of R we have one $U(1)$ gauge symmetry carried by the left-movers and one $U(1)$ gauge symmetry carried by the right-movers. When $R = \sqrt{\alpha'}$ we get four additional gauge fields. Note that these additional gauge fields carry momentum and winding quantum numbers. That is, they are electrically charged under the $U(1)_L \times U(1)_R$ gauge group. The spacetime interpretation is that we have an enhanced non-abelian gauge symmetry. If you look at the quantum numbers a little more carefully, you can see that it's enhanced from $U(1)_L \times U(1)_R$ to $SU(2)_L \times SU(2)_R$.

T-duality

Another remarkable feature of the spectrum: the mass-shell and level-matching conditions are invariant under exchange of n and w together with $R \to \alpha'/R$. That is, you are free to invert the radius of the circle provided you interchange what you mean by momentum and winding modes. *String theory can't tell the difference between a circle of radius R and a circle of radius α'/R.* This symmetry is known as T-duality.

A few comments: T-duality is a uniquely stringy symmetry, since it relies on having winding modes. It relates spacetime manifolds which are geometrically distinct in the usual metric sense. If you like, "string geometry" would have you identify two manifolds which "particle geometry" would tell you are distinct. This identification means we can restrict the radius of the circle to $\sqrt{\alpha'} \leq R \leq \infty$. The special radius at which we had enhanced gauge symmetry, $R = \sqrt{\alpha'}$, is invariant under T-duality.

Compactification on higher-dimensional tori

Suppose we periodically identify a bunch of dimensions.

$$x^I \approx x^I + 2\pi R_I \qquad I = 26 - d, \ldots, 25$$

The analysis goes through pretty much as in the case of circle compactification. The left- and right-moving momenta are

$$\begin{aligned} p_L^I &= \frac{n_I}{R_I} + \frac{w_I R_I}{\alpha'} \\ p_R^I &= \frac{n_I}{R_I} - \frac{w_I R_I}{\alpha'} \end{aligned}$$

where n_I, w_I are the momentum and winding numbers. The mass-shell condition is

$$\begin{aligned} m^2 &= \frac{1}{2}(p_L^2 + p_R^2) + \frac{2}{\alpha'}(N + \tilde{N} - 2) \\ &= \sum_I (n_I/R_I)^2 + \sum_I (w_I R_I/\alpha')^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2) \end{aligned}$$

Level-matching requires

$$p_L^2 - p_R^2 = \frac{4}{\alpha'}(\tilde{N} - N)$$

or equivalently

$$n \cdot w = \tilde{N} - N\,.$$

The analysis of the spectrum is similar to the circle case. For example, you get an enhanced gauge symmetry whenever one of the radii is equal to $\sqrt{\alpha'}$.

This all seems straightforward, but if we dress things up a bit we've actually stumbled on some remarkable mathematics. Define rescaled momenta

$$k_L^I = \sqrt{\frac{\alpha'}{2}}\, p_L^I$$
$$k_R^I = \sqrt{\frac{\alpha'}{2}}\, p_R^I$$

The possible momentum and winding numbers define a lattice in $\mathbb{R}^{2d}$.

$$\Gamma = \left\{(k_L^I, k_R^I)\right\} = \left\{\frac{1}{\sqrt{2}}\left(\frac{n_I\sqrt{\alpha'}}{R_I} + \frac{w_I R_I}{\sqrt{\alpha'}},\ \frac{n_I\sqrt{\alpha'}}{R_I} - \frac{w_I R_I}{\sqrt{\alpha'}}\right)\right\} \subset \mathbb{R}^{2d} \tag{15}$$

The mass-shell condition reads

$$\begin{aligned}\alpha' m^2/2 &= \frac{1}{2}(k_L^2 + k_R^2) + N + \tilde{N} - 2 \\ &= \frac{1}{2}\|k\|^2 + N + \tilde{N} - 2\end{aligned}$$

where we've introduced a Euclidean norm $\|\cdot\|$ on $\mathbb{R}^{2d}$. The level-matching condition reads

$$(k,k) \equiv k_L^2 - k_R^2 = 2(\tilde{N} - N)$$

where we've introduced a Lorentzian inner product $(\cdot,\cdot)$ on $\mathbb{R}^{d,d}$. The string spectrum is encoded in properties of the lattice. For example, we get enhanced gauge symmetry from points k in the lattice with $\|k\|^2 = 2$ and $(k,k) = \pm 2$. (The sign of the Lorentzian norm determines whether the gauge symmetry is carried by the left-movers or right-movers).

The lattice we have constructed satisfies some remarkable properties. A few definitions:

1. A lattice Γ is *even* if $(k,k) \in 2\mathbb{Z}$ for all $k \in \Gamma$.
2. Given a lattice Γ, the *dual lattice* Γ^* is the set of points $x \in \mathbb{R}^{d,d}$ for which

   $$(x,k) \in \mathbb{Z} \qquad \forall k \in \Gamma\,.$$

   A lattice is *self-dual* if $\Gamma^* = \Gamma$.

It's easy to see that the lattice we have defined in (15) is even, since

$$(k,k) = 2n \cdot w \in 2\mathbb{Z}\,.$$

With a little more work it's not hard to show that it's also self-dual. These properties are not accidents: we need an even lattice so that we can satisfy the level-matching condition, and a self-dual lattice is necessary for modular invariance (Polchinski, p. 252).

This is the general story: *toroidal compactification of d spatial dimensions is defined by taking $(p_L, p_R) \in \sqrt{\frac{2}{\alpha'}}\,\Gamma$ where Γ is an even, self-dual lattice in $\mathbb{R}^{d,d}$.* This makes things sound more mysterious than they really are; in the same way that we constructed an ESD lattice above, starting from string theory compactified on a product of circles, you can show that the general ESD lattice arises from compactification on a torus with general constant values for g_{IJ} and b_{IJ}. See Narain, Sarmadi, Witten *Nucl. Phys.* **B279** (1987) 369.

Let's see if we can construct all ESD lattices. One is easy: defining a $2d$-component vector $\binom{n_I}{w_I}$ gives an ESD lattice, where the Lorentzian metric is

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\,.$$

This is exactly the lattice we would get by setting all radii equal to $\sqrt{\alpha'}$ in (15) and switching to light-front coordinates. We'll denote this lattice $\Gamma^{d,d}$. Clearly we can make other ESD lattices by applying $O(d,d,\mathbb{R})$ transformations, setting

$$\Gamma = E\Gamma^{d,d} \qquad E \in O(d,d,\mathbb{R})\,. \tag{16}$$

Remarkably, one can obtain *all* ESD lattices in this way. If you like, you can think of $E = (e_1, \ldots, e_{2d})$ as a matrix built from a set of basis vectors for the lattice.

Now let's see if we can identify the space of inequivalent toroidal compactifications. At first, you might think the space of compactifications is a copy of $O(d,d,\mathbb{R})$. But that would be overcounting: there might be more than one set of basis vectors that generate exactly the same lattice. That is, you might be able to make a transformation

$$e_i \to e_i' = e_j M^j{}_i \qquad i,j = 1,\ldots,2d$$

such that $\{e_i\}$ and $\{e'_i\}$ generate the same lattice.[1] In terms of the matrix E the change of basis is

$$E \to EM$$

This means not all $E \in O(d,d,\mathbb{R})$ generate inequivalent lattices. Rather we need to identify

$$E \sim EM \qquad M \in O(d,d,\mathbb{Z})$$

where $O(d,d,\mathbb{Z})$ denotes the subgroup of $O(d,d,\mathbb{R})$ with integer entries.[2] This means the space of inequivalent lattices is the space of left cosets, $O(d,d,\mathbb{Z})\backslash O(d,d,\mathbb{R})$.[3] We're still overcounting, though, because some of these lattices lead to identical string theories. In particular the string spectrum (the mass-shell condition) is invariant under $O(d,\mathbb{R}) \times O(d,\mathbb{R})$ transformations that act separately on p_L and p_R. So we also need to identify

$$E \sim XE \qquad X \in O(d,\mathbb{R}) \times O(d,\mathbb{R})\,.$$

Thus the "Narain moduli space" of inequivalent toroidal compactifications is the double quotient

$$O(d,d,\mathbb{Z})\backslash O(d,d,\mathbb{R})/\left(O(d,\mathbb{R}) \times O(d,\mathbb{R})\right)\,.$$

A few comments:

1. Although this expression for the moduli space is accurate (I hope), there's some redundancy in this description, since some of the symmetries contained in $O(d,\mathbb{R}) \times O(d,\mathbb{R})$ are also contained in $O(d,d,\mathbb{Z})$.

2. $O(d,d,\mathbb{Z})$ contains a $\mathbb{Z}_2$ subgroup $\{\mathbb{1}, -\mathbb{1}\}$ that acts in kind of a stupid way on the lattice. So we write

$$O(d,d,\mathbb{Z}) = \mathbb{Z}_2 \times PO(d,d,\mathbb{Z})$$

and identify the T-duality group with $PO(d,d,\mathbb{Z})$, the "projective orthogonal group" in which the elements M and $-M$ of $O(d,d,\mathbb{Z})$ are identified. See Giveon, Porrati and Rabinovici, hep-th/9401139 section 2.4 and footnote 7 on p. 27.

[1] For example $\left\{\binom{1}{0}, \binom{0}{1}\right\}$ and $\left\{\binom{1}{0}, \binom{1}{1}\right\}$ generate the same lattice inside $\mathbb{R}^2$. In general the group of basis transformations is $GL(2d,\mathbb{Z})$.

[2] $O(d,d,\mathbb{Z})$ is the subgroup of $GL(2d,\mathbb{Z})$ that is also contained in $O(d,d,\mathbb{R})$.

[3] This terminology never made much sense to me. Since the subgroup is acting from the right, shouldn't it be called a space of right cosets and denoted $O(d,d,\mathbb{R})/O(d,d,\mathbb{Z})$?