Phys V3500/G8099
homework #1
due Thurs. Jan. 31

1. **Saddle-point evaluation**

   Consider the following "action" for an N-component vector $\mathbf{x}$.

$$S = \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{b}\cdot\mathbf{x}$$

   Here A is a real symmetric $N \times N$ matrix and $\mathbf{b}$ is an N-component vector.

   (i) Solve the "classical equations of motion" $\frac{\partial S}{\partial \mathbf{x}} = 0$ to find the "classical trajectory" $\mathbf{x}_{\text{cl}}$.

   (ii) Set $\mathbf{x} = \mathbf{x}_{\text{cl}} + \mathbf{y}$ and write the action in terms of $\mathbf{y}$.

   (iii) Evaluate the "path integral" $\int \mathcal{D}\mathbf{y} e^{iS}$ (it doesn't matter whether you integrate over $\mathbf{x}$ or $\mathbf{y}$). Express your answer in terms of the action for the classical trajectory and the determinant of (some constant times) A. If you're worried about convergence you can give A a small positive imaginary part.

2. **Path integral for a free particle**

   Consider a free non-relativistic particle moving in one dimension. The Hamiltonian is just $H = p^2/2m$.

   (i) Compute $G(x_f, t_f | x_i, t_i) = \langle x_f | e^{-iH(t_f - t_i)} | x_i \rangle$ using ordinary methods, for example by inserting a complete set of momentum eigenstates.

   (ii) Write down the path integral expression for G.

   (iii) As in problem 1, find the classical trajectory $x_{\text{cl}}(t)$: that is, the trajectory that satisfies the classical equations of motion with appropriate boundary conditions. Set $x(t) = x_{\text{cl}}(t) + y(t)$ and rewrite the path integral in terms of y.

   (iv) Formally evaluate the Gaussian path integral over y.

(v) In the path integral formulation, note that the dependence on x_i and x_f comes purely from the classical action. Does it agree with the x_i and x_f dependence you found in part (i)?

(vi) By matching your answers in parts (i) and (iv) obtain a formula of the form

$$\det{}^{-1/2}(\text{some operator}) = \text{something}$$

What is the operator? What space of functions does it act on? And what expression shows up on the right hand side?

3. **Stress tensor for a point particle**

Start with the Polyakov action for a relativistic particle in Minkowski space.

$$S = \int d\tau \, \frac{1}{2e} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \frac{1}{2} e m^2$$

Notation: $\eta_{\mu\nu}$ is the Minkowski metric, x^μ are spacetime coordinates, and $X^\mu(\tau)$ is the particle worldline.

(i) Derive the *worldline* conserved quantities P^μ associated with invariance of the action under $X^\mu \to X^\mu + \text{const}$. Use the equation of motion for e to express P^μ purely in terms of $\dot{X}^\mu$.

(ii) Although you derived a worldline conservation law $\frac{\partial}{\partial\tau} P^\mu = 0$, at least for timelike worldlines this can also be interpreted as a spacetime conservation law $\frac{\partial}{\partial t} P^\mu = 0$. Show that you can write $P^\mu = \int d^3x \, T^{0\mu}(t, \mathbf{x})$ in terms of the "stress tensor"

$$T^{\mu\nu}(x) = \int d\tau \, \frac{1}{e} \dot{X}^\mu \dot{X}^\nu \delta^4(x - X(\tau)) \tag{1}$$

See e.g. Weinberg, Gravitation and cosmology, section 2.8.

(iii) Consider the Polyakov action for a particle in a curved spacetime with metric $g_{\mu\nu}$.

$$S = \int d\tau \, \frac{1}{2e} \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}(X) - \frac{1}{2} e m^2$$

Show that the obvious generalization of (1) to curved space satisfies

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2} T^{\mu\nu} \, .$$