Phys V3500/G8099
homework #2
due Thursday Feb. 7

1. **Measure on space of constant metrics**

   Suppose we had an n-dimensional manifold M with coordinates x^i and metric $g_{ij}(x)$. The metric determines an inner product on the tangent spaces, for example
   $$\delta x,\, \delta y \in T_x M \quad \Rightarrow \quad (\delta x, \delta y) = g_{ij}\delta x^i \delta y^j \,.$$
   It also determines a measure for integrating over M, namely $\int d^n x \, \sqrt{\det g_{ij}(x)}$. A cute way of relating these two quantities is to note that
   $$\int d^n(\delta x) e^{-(\delta x, \delta x)} \sim \frac{1}{\sqrt{\det g(x)}}$$
   i.e. a Gaussian integral over one of the tangent spaces generates the (inverse of) the integration measure.

   Let's apply this procedure to determine the right measure for integrating over the space of constant metrics on the interval $[0, 1]$. The space of constant metrics is just
   $$\mathbb{R}^+ \simeq \left\{ s > 0 \, : \, s = \int_0^1 d\tau \sqrt{g_{\tau\tau}} \right\} .$$
   Any point in this space corresponds to a metric $g_{\tau\tau} = e^2$ where the einbein $e(\tau) = s$ is a constant (independent of τ). For a given s, consider a small constant fluctuation in the metric $g_{\tau\tau} \to g_{\tau\tau} + \delta g_{\tau\tau}$. Define a covariant norm on these metric fluctuations
   $$(\delta g, \delta g) = \int_0^1 d\tau \sqrt{g_{\tau\tau}} g^{\tau\tau} g^{\tau\tau} \delta g_{\tau\tau} \delta g_{\tau\tau}$$

   (i) What is the corresponding norm on constant fluctuations in the einbein? Determine it as explicitly as possible, i.e. perform the τ integral and write your answer in terms of δe and s.

   (ii) By considering the Gaussian integral $\int d(\delta e) e^{-(\delta e, \delta e)}$ argue that, up to an overall constant, the right measure for integrating over the space of constant metrics is $\int ds/\sqrt{s}$.

2. **Measure on moduli space**

Any metric on the interval $[0, 1]$ can be expressed as a diffeomorphism acting on a constant metric. This means we can write

$$\int \mathcal{D}(\text{einbein}) = \int \mathcal{D}(\text{constant einbein})\mathcal{D}(\text{diffeomorphisms}) \det \frac{\delta(\text{einbein})}{\delta(\text{diffeomorphism})}$$

and therefore

$$\int \frac{\mathcal{D}(\text{einbein})}{\text{diffeomorphsims}} = \int \mathcal{D}(\text{constant einbein}) \det \left.\frac{\delta(\text{einbein})}{\delta(\text{diffeomorphism})}\right|_{\text{einbein} = \text{const.}}$$

We already know the measure for integrating over constant einbeins; now let's see if we can figure out the determinant.

(i) An infinitesimal diffeomorphism corresponds to a vector field ϵ^τ with the boundary conditions $\epsilon^\tau(0) = \epsilon^\tau(1) = 0$. Assuming $g_{\tau\tau} = (e_\tau)^2$ is constant, under such a diffeomorphism $\delta g_{\tau\tau} = 2 g_{\tau\tau} \partial_\tau \epsilon^\tau$. Use this to argue that $\delta e^\tau = \frac{1}{s}\frac{\partial}{\partial\tau}\epsilon^\tau$ and therefore

$$\left.\frac{\delta(\text{einbein})}{\delta(\text{diffeomorphism})}\right|_{\text{einbein} = \text{const.}} = \frac{1}{s}\frac{\partial}{\partial\tau}.$$

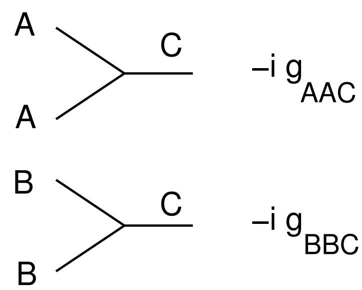
What space of functions does ∂_τ act on?

(ii) Up to a constant, what is the determinant? (It's simplest to rescale τ and relate it to the determinant you saw on the last homework.)

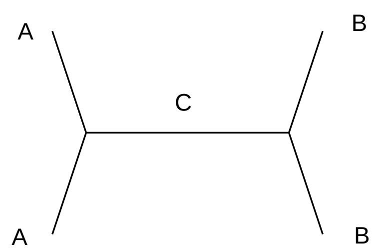
(iii) Putting your results in problems 1 and 2 together, argue that the right measure for integrating over the moduli space of $\{\text{metrics}\}/\{\text{diffeomorphisms}\}$ is just $\int_0^\infty ds$!

3. **Scattering**

Consider two distinct species of massless scalar particles A and B which can interact with a third scalar C of mass m. The worldlines can join or split with amplitude



(i) Compute the amplitude $\mathcal{M}$ for $AA \to BB$ scattering from the diagram



Express your answer in terms of the center of mass energy E_{cm} and scattering angle θ.

(ii) Suppose the A and B particles only interact gravitationally. Compute the amplitude for $AA \to BB$ scattering from the diagram