Phys V3500/G8099
homework #4
due Thursday Feb. 21

1. **Vertex operators**

   To compute string scattering amplitudes we introduced vertex operators $e^{ik \cdot X}$. But what do these vertex operators have to do with the intuitive picture of strings coming in from infinity, scattering, then separating? If you've been worried, this exercise should put your mind at ease.

   (i) Convince yourself that on the Riemann sphere the Green's function you derived, namely
   $$G(x) = \frac{1}{2\pi} \log \mu |x|$$
   secretly has, besides the unit positive source you wanted at the origin, a unit negative source at infinity. A conformal map $z \to 1/z$ is probably helpful.

   (ii) Consider the string path integral with two vertex operators, one at the origin and one at infinity.
   $$\int \mathcal{D}X^\mu(\cdot)\, e^{ik \cdot X(0)} e^{-ik \cdot X(\infty)} \exp \left\{ -\frac{1}{4\pi\alpha'} \int d^2\sigma\, \delta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \right\}$$
   Note that I've imposed momentum conservation, but I haven't worried about $SL(2,\mathbb{C})$. If you like, you could think of this path integral as a piece of a larger calculation.

   Anyhow: use your Green's function to find the classical saddle point of this path integral. Work in complex coordinates z, $\bar{z}$ on the worldsheet and be sure to include the vertex operators in identifying your saddle point.

   (iii) Set $z = e^w$ where $w = \tau + i\sigma$ with $-\infty < \tau < \infty$, $\sigma \sim \sigma + 2\pi$. Re-express your classical worldsheet in terms of τ and σ.

   (iv) Un-do the Wick rotation by setting $\tau = i\tau_M$ to return to a Lorentzian worldsheet with coordinates τ_M, σ. What does your classical worldsheet look like in terms of τ_M and σ?

2. **Unitarity**

Recall the integral representation of the Virasoro amplitude

$$\mathcal{M}(k_1, k_2, k_3, k_4) = \int d^2z \, |z|^{\alpha' k_1 \cdot k_4} |1 - z|^{\alpha' k_2 \cdot k_4} \,. \tag{1}$$

Let's see how this amplitude behaves at high energies with fixed scattering angle. In terms of Mandlestam invariants we're interested in the regime

$$s \to \infty \qquad t, u \to -\infty \qquad t/s \text{ and } u/s \text{ fixed}$$

(i) Start by rewriting (1) in terms of s, t, u rather than $k_1 \cdot k_4$ and $k_2 \cdot k_4$.

(ii) Evaluate the integral over z using a saddle point approximation. That is, find the value of z for which the integrand is stationary and evaluate the integrand at that point. You can make approximations appropriate to the kinematics of interest, in particular $s + t + u \approx 0$.

(iii) Use your result in (ii) to express $|\mathcal{M}|$ in this regime in terms of $s = 4E^2$ and the center of mass scattering angle θ.

Moral of the story: even though they include graviton exchange, string theory amplitudes are extremely well behaved at high energies and have no problems with unitarity. Reference: Gross and Mende, *Phys. Lett.* **B197** (1987) 129.