Phys V3500/G8099
homework #8
due Friday April 25

1. **Compactification on S^1 revisited**

   The purpose of this exercise is to reproduce our results on S^1 compactification from the lattice point of view.

   (i) Show that
   $$\left\{ \frac{1}{\sqrt{2}} \left(\frac{n\sqrt{\alpha'}}{R} + \frac{wR}{\sqrt{\alpha'}}, \; \frac{n\sqrt{\alpha'}}{R} - \frac{wR}{\sqrt{\alpha'}} \right) \right\}$$
   is an even, self-dual lattice in $\mathbb{R}^{1,1}$ with metric $(+-)$.

   (ii) How does the group $O(1,1,\mathbb{R})$ act on the lattice?

   (iii) Identify the subgroup $O(1,1,\mathbb{Z}) \subset O(1,1,\mathbb{R})$ and show how it acts on the lattice.

   (iv) Identify the T-duality group $PO(1,1,\mathbb{Z})$ and show how it acts on the lattice.

   (v) What is the moduli space
   $$O(1,1,\mathbb{Z}) \backslash O(1,1,\mathbb{R}) / (O(1,\mathbb{R}) \times O(1,\mathbb{R}))\,?$$

2. **The $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$ heterotic strings**

   To construct the heterotic string theories we take the left-moving modes of the bosonic string
   $$X_L^\mu(\tau + \sigma) \qquad X_L^I(\tau + \sigma)$$
   and marry them to the right-moving modes of the superstring
   $$X_R^\mu(\tau - \sigma) \qquad \tilde{\psi}^\mu(\tau - \sigma)\,.$$
   Here the indices $\mu, \nu = 0, \ldots, 9$ and $I, J = 10, \ldots, 25$. The additional dimensions of the bosonic string are compactified on a torus characterized by an even self-dual lattice $\Gamma \subset \mathbb{R}^{16}$. (Since we're only compactifying left-movers, $\mathbb{R}^{16}$ carries a Euclidean norm.)

There are exactly two ESD lattices in $\mathbb{R}^{16}$. One is denoted $\Gamma^8 \oplus \Gamma^8$, where Γ^8 is an ESD lattice in $\mathbb{R}^8$. An (overcomplete) basis for Γ^8 is

$$\begin{aligned} &\pm e_i \pm e_j && i, j = 1, \ldots, 8 \\ &\pm \frac{1}{2} e_1 \pm \frac{1}{2} e_2 \cdots \pm \frac{1}{2} e_8 && \text{even number of + signs} \end{aligned}$$

Here e_i are orthonormal basis vectors and the $\pm$ signs are chosen independently except for the restriction stated explicitly in the second line. The other ESD lattice in $\mathbb{R}^{16}$ is denoted Γ^{16}. The definition of Γ^{16} looks just like the definition of Γ^8, except that you work in $\mathbb{R}^{16}$ and the indices take values $i, j = 1, \ldots, 16$.

Your assignment: work out the spectrum – in particular the Lorentz quantum numbers – of the massless states of the two heterotic string theories.

A few comments:

(i) You need to impose the Virasoro constraints. The important ones are the zero modes

$$\begin{aligned} &\frac{1}{4}\alpha' \left(p_\mu p^\mu + p_L^I p_L^I\right) + N - 1 = 0 && \text{on the left-movers} \\ &\frac{1}{4}\alpha' p_\mu p^\mu + \tilde{N} - \frac{1}{2} = 0 && \text{right-movers, NS sector} \\ &\frac{1}{4}\alpha' p_\mu p^\mu + \tilde{N} = 0 && \text{right-movers, R sector} \end{aligned}$$

(ii) You need to make the GSO projection on the right-movers, $(-1)^{\widetilde{F}} = 1$ where the operator $\widetilde{F}$ is defined in Polchinski (10.2.24). A somewhat more tractable expression for $(-1)^{\widetilde{F}}$ can be found in GSW on p. 219 (NS sector) and p. 222 (R sector).

A comment: the gauge fields you found with $p_L = 0$ generate the Cartan subalgebra (the maximal abelian subalgebra) of the full gauge group. The remaining gauge fields fill out the adjoint representation of either $E_8 \times E_8$ or $SO(32)$.

3. **T-duality of type II**

Consider the IIA string compactified on a circle, with $X^9 \approx X^9 + 2\pi R$. Just as in the bosonic string, we define T-duality as a one-sided parity transformation

$$X_R^9 \rightarrow -X_R^9 \qquad \widetilde{\psi}^9 \rightarrow -\widetilde{\psi}^9 \,.$$

Show that this transformation gives you the IIB string on a circle of radius α'/R. (The main thing is to show that the IIA GSO projection gets mapped to the IIB GSO projection. On p. 26 Polchinski denotes these by

$$\begin{array}{ll} \text{IIB}: & (NS+, NS+) \quad (R+, NS+) \quad (NS+, R+) \quad (R+, R+) \\ \text{IIA}: & (NS+, NS+) \quad (R+, NS+) \quad (NS+, R-) \quad (R+, R-) \end{array}$$

If you need some help with how parity acts on spinors see Peskin and Schroeder p. 65.)

A comment: unlike in the bosonic string, there is no minimum radius for a circle in type IIA. You have to allow $0 < R < \infty$ to get all possible inequivalent compactifications of type IIA. What you've shown is that IIA on a circle of radius $R < \sqrt{\alpha'}$ can be reinterpreted as IIB on a circle with $R > \sqrt{\alpha'}$.