A.C. CIRCUITS

Introduction

Consider the circuit in Figure 1 consisting of a resistance \( R \), a capacitance \( C \), and an inductance \( L \), connected in series and driven by a time-varying voltage source.

\[
\int \frac{di}{dt} + \frac{q}{C} + iR + \frac{1}{C} \int i dt = 0
\]

(1)

We restrict our attention to driving voltages that have a sinusoidal dependence on time:

\[
v = V_m \sin(\omega t + \phi).
\]

(2)

With \( V_m \) the amplitude of the driving AC signal. The angular frequency \( \omega \) (in rad / s) is related to the oscillation period \( T \) by \( \omega = \frac{2\pi}{T} \). Remember that the phase angle (\( \phi \)) in the above equation is arbitrary; if it is \( \pi/2 \), then the dependence turns into a cosine function. You saw in 1402 (hopefully !) that the solution to Eq. 1 gives a sinusoidally varying current,

\[
i = I_m \sin(\omega t + \phi').
\]

(3)
where $I_m$ is the amplitude of the current moving in the circuit and $\phi'$ is a phase factor which, in general, differs from the phase of the driving signal. *(note: In our notation, $i$ and $v$ represent instantaneous time-varying functions, while $I_m$ and $V_m$ are amplitudes and, thus, time independent).* There are also transient solutions to (1) which have more complicated behavior with time. They are associated with turning the circuit on or off, and they usually decay quickly. We will ignore these transients in this lab.

The voltage across a resistor is given by the first term in (1) as:

$$v_R = iR = R I_m \sin \omega t .$$

(4)

Thus, for a resistor the voltage is *in phase with the current*. Note that if a circuit contained only a resistor then we would obtain $I_m = V_m / R$, irrespective of the frequency of the driving signal. In general, however, the relationship between the current amplitude and voltage amplitude is more complicated and is frequency dependent. We express this relationship in terms of the frequency-dependent *impedance* of the circuit, $I_m = V_m / Z(\omega)$. For the case of a resistor,

$$Z_R = R .$$

Next, the voltage across an inductance is:

$$v_L = L \frac{di}{dt} = \omega L I_m \cos \omega t$$

(5)

$$= \omega L I_m \sin(\omega t + \frac{\pi}{2})$$

The voltage is again a sinusoidal function of $t$. However, unlike the case of a resistor, the time dependence of the voltage is shifted by $\pi / 2$ relative to the current--the voltage is $90°$ *out of phase* with the current.

From (5) we see that

$$Z_L = \omega L .$$

(6)

Note that the impedance of an inductance depends linearly on the *frequency*.

Finally, the voltage across a capacitance is

$$v_C = \frac{1}{C} \int i dt$$

$$= \frac{1}{C} \left( -\frac{I_m}{\omega} \cos \omega t \right)$$

$$= \frac{1}{\omega C} I_m \sin(\omega t - \frac{\pi}{2})$$

(7)
The voltage is again 90° out of phase with the current, but the phase shift is of the opposite sign. The voltage across a capacitance \textit{lags} the current by 90°, whereas we have seen that the voltage across an inductance \textit{leads} the current by 90°.

From (7) we see that

$$Z_C = \frac{1}{\omega C}.$$  \hspace{1cm} (8)

We now find the overall impedance $Z_{\text{total}}$ of the series RLC circuit:

$$Z_{\text{total}} = \frac{V_m}{I_m},$$  \hspace{1cm} (9)

and we find the phase difference $\phi$ between the driving voltage $v$ and the current $I$, for given values of $R$, $L$, $C$, and $\omega$. Equation (2) for $v$ can be rewritten as:

$$v = V_m \sin(\omega t + \phi) = V_m(\sin \omega t \cos \phi + \cos \omega t \sin \phi).$$  \hspace{1cm} (10)

Equation (1) for $v$ can be rewritten as:

$$v = v_R + v_L + v_C$$

$$= (RI_m \sin \omega t) + (\omega L I_m \cos \omega t) - \left(\frac{1}{\omega C} I_m \cos \omega t\right)$$  \hspace{1cm} (11)

Since expressions (10) and (11) must be equal for all times $t$, we may equate coefficients of $\sin \omega t$ and of $\cos \omega t$:

$$V_m \cos \phi = RI_m$$

$$V_m \sin \phi = \left(\omega L - \frac{1}{\omega C}\right) I_m$$  \hspace{1cm} (12)

Hence the phase $\phi$ is given by:

$$\tan \phi = \frac{\omega L - \frac{1}{\omega C}}{R}.$$  \hspace{1cm} (13)

Eliminating $\phi$ in Eqs. (12) by squaring the two equations and adding them together, we obtain:

$$V_m = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 I_m}.$$
The total impedance of the series circuit is then
\[ Z_{\text{total}} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}. \] \hspace{1cm} (14)

An impedance that produces a positive 90° phase shift is alternatively called the **reactance** \( X \) of the element. So we have
\[ X_L = \omega L \]
\[ X_C = -\frac{1}{\omega C}. \]

The combined reactance of the two elements in series is
\[ X = X_L + X_C = \omega L - \frac{1}{\omega C}. \]

We see from (13) that \( \tan \phi = \frac{X}{R} \) and from (14) that \( Z_{\text{total}} = \sqrt{R^2 + X^2} \). These relationships suggest a convenient graphical representation. In Figure 2-a, the impedance is shown as a vector whose components are the resistance \( R \) and the reactance \( X \). The angle between the vector and the horizontal axis is the phase angle \( \phi \).

The voltage may be presented on a similar plot. Merely by multiplying all impedances by \( I_m \), we are led to Figure 2-b.

Note that the impedance of the circuit is minimum when \( \omega L = \frac{1}{\omega C} \). Also note that the impedance would go to zero in the physically impossible case of zero resistance. The strong response of the circuit (i.e. the large resulting current flow) at a particular frequency is called a resonance. The resistance of the circuit regulates the behavior of the circuit so that the current response remains finite. It also widens the frequency range over
which the “resonance peak” is spread. You will study this aspect of the response of an AC circuit in your lab.
Procedure

The first thing you will want to do is study your driving signal on the oscilloscope. Set the signal generator to the sinusoid setting (~). Connect the output of the signal generator to one of the scope probes (e.g. on channel 1) and adjust the trigger on the oscilloscope until you see the signal. It will be easiest for you if you set the “coupling” on the input and on the trigger to “AC”. Then adjust the frequency setting on the signal generator to 10 kHz (a good starting point) and adjust the time base on the oscilloscope so that you can see a full cycle of the wave. Fine-tune the frequency until you get exactly 10 kHz (one cycle in 100 µs). Next, adjust the voltage amplitude of the signal generator to 1 V. You should then have a signal source described by

\[ V = (1 V) \sin((2\pi \cdot 1 \times 10^4) t + \phi). \]

You can effectively set the phase to zero by adjusting the horizontal position on the oscilloscope so that the voltage is at zero at one of the division crossings on the scope. That division crossing then becomes your “zero” time. For most of the lab you will need both channels on the scope so you will need to set the scope up to use the external trigger into which you will provide the same signal generator output that you have already been looking at. The “external trigger” behaves identically to the trigger on one of the input channels – it causes the signal to be displayed when the external signal input crosses some threshold. Set up the external trigger input from the signal generator while keeping the same signal on channel 1. Adjust the trigger threshold so that the signal on channel 1 starts at zero. Adjust the horizontal position of the signal to place the start at your chosen “zero time” division crossing. Now you have essentially calibrated the timing of the external trigger. Leave the trigger threshold unchanged for the remainder of your lab – if you change it you will need to re-time the external trigger.

A. Frequency Dependence of the Impedance of L and C

In this part of the lab you will measure the frequency dependence of the impedances \( Z_L \) and \( Z_C \) (Eqs. 6 and 8) by measuring \( V_L \) and \( V_C \), which will be proportional to the impedances if \( I_m \) is kept constant. (See Eqs. 5 and 7). However, as the frequency of the signal generator is varied, not only does the impedance of L or C change, but also the impedance of the circuit as a whole changes (Eq. 14). Hence if \( V_m \) (the amplitude of the signal generator’s output voltage) is left constant when the frequency is changed, the \( I_m \) will also change (Eq. 9). To keep \( I_m \) constant, adjust \( V_m \) to keep the voltage across R constant, as \( V_R \) is proportional to \( I_m \).

To start, connect the resistor, inductor, and capacitor that the TA gives you into a series circuit like that shown above on your breadboard with the signal generator connected across the three components. Accurately measure the resistance of the resistor before connecting up the circuit and record the value in your logbook. Connect one scope probe across the resistor and plug this signal into channel 1. You will use this signal to adjust the amplitude of the voltage source to keep the amplitude of current flow through the resistor constant. The automatic readout of amplitude on the oscilloscope will help you greatly in performing this task. Now, connect another set of scope probes across the capacitor and
plug them into channel 2 of the oscilloscope. One this signal you are directly measuring
the voltage across the capacitor. The ratio of the amplitudes of the signal on channel 2 to
channel 1 should be given by

\[ V_c = \frac{V_R}{R} Z_C(\omega). \]

Use this equation to measure the impedance of the capacitor at 10 kHz. Now vary the
frequency from 1 kHz to 100 kHz and measure the impedance of the capacitor over this
frequency range. Use the automatic frequency read-out on the oscilloscope to help you
adjust the signal generator. Make at least 10 measurements at suitable frequencies such
that you have relatively evenly spaced points on the frequency axis.

Now do the same set of measurements for the inductor. Choose the same frequency
settings as for the capacitor.

Plot the results (choosing variables for ordinate and abscissa with some forethought) so as
to make a simple graphical comparison with Eqs. (5) and (7). Note: if \( y = kx \), then the
graph of \( y \) vs. \( x \) should be a straight line. If \( y = k/x \), then the graph of \( y \) vs. \( 1/x \) should be
a straight line -- see Figure 3.
Check the expected relationships between frequency and impedance. In practice, the elements may not be pure R, L, and C. The resistor may have an associated inductance, the inductance may have an appreciable resistance, etc. Do your curves show any evidence of these effects (do they show any systematic deviations from the expected behavior)? Is there any danger that connecting the oscilloscope will alter the behavior of the circuit (the scope has an input resistance of 1 MΩ, with an input capacitance, in parallel, of 35 pF)? Analyze the circuit taking into account the capacitance and resistance of the scope input and quantitatively evaluate the effect of the scope on your measurements. Include this analysis in your write-up.

B. Voltages in the Series RLC Circuit: Magnitudes and Phase Shifts

In this part of the experiment, you will measure the phase shifts of the signals induced across the capacitor and inductor and of the current flow in the circuit as a whole. For these measurements you will first compare the phases of the current in the circuit and the signals across the components. Then you will measure the phase shifts between the current flow in the circuit and the driving voltage.

1. For the first set of measurements you will want to set the scope back to trigger on channel 1 which should still be the voltage measured across the resistor which is proportional to the current flowing through the resistor. You will then look at the voltage across the components on channel 2 which will be shifted with respect to the current. We know (and are relying on the fact) that the current and voltage across the resistor are in phase. Connect the probes on channel 2 across the capacitor and look at the two traces on the scope. If you adjust the horizontal scale you should be able to estimate the fraction of a cycle by which the voltage and current are shifted in the capacitor. Verify that you get the expected magnitude and direction of phase shift. Plot the two traces in your log book and show them in your write-up. Do the same for the inductor.

2. For the second set of measurements, connect the probe on channel 1 to the signal generator and the probe on channel 2 across the resistor. Now you can directly measure the phase difference between the current flowing in the circuit and the driving voltage. Because this phase difference takes on a whole range of values it may be easiest for you to measure the phase difference using the horizontal cursors to measure the time difference between zero-crossings of the voltage and current and calculate the phase difference,

$$\Delta \phi = \frac{\Delta t}{\omega} = \frac{\Delta t}{2\pi f}.$$  

Measure the phase difference as a function of frequency over the same range of frequencies given above. Remember that a positive phase shift gives a left or negative shift of the curve on the time axis. Compare your results quantitatively to the expectations from Eq. 13.
C. Resonance in a Series RLC Circuit

As noted above, the series RLC circuit has a resonance, i.e., there is a frequency where the impedance $Z_{\text{total}}$ (see Eq. 14) takes on a minimum value, so the current takes on a maximum value for a given driving voltage. The resonance will be sharpest (and therefore easiest to see) for small $R$, large $L$, and small $C$, although the position of the resonance depends only on $L$ and $C$. Use the adjustable resistor on your breadboard (1 kΩ max) and Set it initially to 100 Ω. Now vary the frequency of the signal generator while observing the current response through the voltage across the resistor. You can start simply by varying the frequency and watching the amplitude of the response until you find a region where it increases and then decreases. Once you have found the resonance region (which you should also check by solving $\omega L = 1/\omega C$) make several measurements of the current response on both sides of the peak. Make enough measurements that you can accurately determine the frequency at which the response peaks. Also make measurements far from the resonance peak to show that the current response of the circuit asymptotically approaches zero. Now, adjust the resistance of the variable resistor to 500 Ω and repeat the measurements of the current response around the peak frequency. You should find a much lower current and a wider peak. Do another set of measurements with the resistance of your variable resistor set to 10 Ω.

The response of a resonance circuit is often written:

$$A = A_0 \frac{(\Gamma^2 / 4)}{(\omega - \omega_0)^2 + \Gamma^2 / 4}.$$  

This form is convenient because when $(\omega - \omega_0) = \Gamma/2$, the response is ½ the maximum. Thus, $\Gamma$ is called the “full-width at half max.” or FWHM for short. Often, $\Gamma$ is simply called the width of the resonance. Estimate the widths of the resonance peaks from your measurements at the three resistance values and compare to what you expect. Might you expect deviations at low $R$? Why? Include in your write-up plots of your resonance peaks and the widths that you have extracted and quantitative analysis of both the measured peak positions and widths in terms of your expectations.