Physics 1494 – General Notes

Introduction

The experiments you will perform this semester have been selected with two purposes in mind: to related physical experience to some of the concepts which you have learned in your previous Physics courses, and to provide an opportunity to learn some of the skills and techniques which physicists use in exploring the connection between theory and experiment. The laboratory instructors will be available to help you understand the basis for each experiment, but the responsibility for preparation lies with you. The amount of learning and enjoyment which you experience depends on your effort.

Significant Figures

Suppose that during the process of an experiment an investigator is confronted with the task of determining the circumference of a pulley. She measures the diameter of the pulley with a good ruler whose smallest division is $1/10$th inch and records the diameter as 4.15 inches. She takes $\pi = 3.14156$, and multiplying out $3.14156 \times 4.15$ writes down: Circumference 13.037640 inches. Arithmetically the result is correct. Physically it is misleading. Confronted with the ruler the experimenter would readily admit that she could not estimate with certainty to one-hundredth of an inch, and that the diameter might well be 4.14 inches rather than 4.15 inches. If, however the diameter of the pulley is taken as 4.14 inches, the circumference would come out according to her calculations 13.006229 inches. Comparing the result with the former she finds that they differ in the fourth figure.

Which answer is correct? The investigator does not know, and in fact using the apparatus mentioned no one would be willing to say. Another observer might estimate the diameter to be 4.16 inches and a third “circumference” would be obtained. Because of this uncertainty the physicist says that all the answers are wrong since they explicitly express an accuracy in the determination of the circumference which is not warranted by the precision in the measurement of the diameter. She says that some of the numbers written down are significant figures and that the rest are meaningless.

How, then, are we to determine the number of significant figures in a result? For use in the analysis of carefully performed experiments there are statistical methods available which allow a prediction to be made of the precision. Returning to our example we assume that the observer might be willing to state that as a result of a number of trials the diameter of her pulley is $4.15 \pm 0.01$ inches (to be read “4.15 plus or minus 0.01 inches”). In other words, she is willing to state that the diameter is correct to about $\pm 0.2\%$. As will be shown later, her determination of the circumference is then only accurate to $\pm 0.2\%$ and should be written 13.04 $\pm 0.03$ inches.

The experimenter should recognize that the accuracy of her result is independent of the position of the decimal point in the result. If she changes the units in which she expresses a result she can in no way alter the accuracy thereby. The term “significant figures” is often used to give this idea; a result accurate to four significant figures would be one in which
the observer relies on the first four figures counting always from the first number at the left, other than leading zeroes. As an illustration, a certain distance is known to be 15.1 meters, accurate to about 1 per cent, that is, to three significant figures, then the same result would be 0.0151 kilometers or $1.51 \times 10^4$ millimeters, using as before three significant figures.

A frequent misunderstanding arises in the case where the last significant figure is 0. A measurement made with a centimeter rule of the diameter of a cylinder might be 7.10 cm. Here the 0 is a significant figure, the observer could have seen if her measurement had been 7.09 or 7.11. A record of 7.1 cm would indicate that she had failed to read as closely as she might have done, but on the other hand a record of 7.10000 cm could not represent the truth of her observation she could not possibly have observed with her ruler what figures occupied the fourth, fifth and sixth places.

**Curve Plotting**

Since you will be required to plot many of your results, the following suggestions will help to make your graphs more useful.

1. The general idea is the same as that used in graphical methods in algebra, with the same conventions in regard to signs and location of points. If the values of the two quantities to be plotted are not already arranged in columns, you will save time by so arranging them before plotting the points on the graph.

2. A scale of units for each axis should in general be chosen that the final curve will practically (but not necessarily) fill the page, though the units must be one which can readily be applied. Neglect of this caution causes much waste of time and also defeats the purpose of the graph. A common difficulty in this respect arises from an attempt to use on decimal graph paper such a scale as ‘one square for each three units. A convenient rule for decimal co-ordinate paper is to let one square equal one, two, or five units or any of these three numbers multiplied by a power of ten (e.g. 0.1, 0.2, 0.5, or 10, 20, 50 etc.)

3. The prominent divisions should be plainly marked on the two axes. Do not mark every square – too many numbers on the axes generally make the graph confusing. The values of the individual readings should not appear on the axes.

4. Each axis should bear a label which includes the unit used.

5. The separate points should be accurately located and marked in such a way as to be distinguishable after the curve is drawn. An encircled dot is especially convenient for plotting points. When it is important to record the uncertainty of each plotted point, horizontal and/or vertical bars are passed through the point, the lengths of which represent the uncertainty of the point with respect to the x and/or y co-ordinates.
6. If several curves are drawn on the same sheet, it is best to represent them in different ways, e.g., broken and solid lines, and to mark the points of each curve by different symbols.

7. If there is a theory concerning the fundamental relationship of the two variables plotted, then a curve corresponding to that hypothesis should be drawn. In many experiments the theory is not well known in advance and the only hypothesis that can be made is that it will yield a smooth curve. The curve should be drawn through the mean of the various points. The curve need not pass through the first and last point. Instead, each point should be considered as accurate as any other point (assuming their experimental uncertainties are the same), so that the curve is drawn with about as many points above the curve as are below it with the ‘aboves’ and ‘belows’ distributed at random along the curve (i.e., not all points above the curve at one end and below at the other end), as shown below in the figure. Each graph should bear a title telling briefly what the curve represents. As an example, suppose that we wish to plot the curve showing the relation between the observed position of an indicator on the end of a spring and the force pulling the spring, and suppose that we have made the following observations:

<table>
<thead>
<tr>
<th>Position of Indicator (cm)</th>
<th>Force on the Spring (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3 ± 0.2</td>
<td>5.00</td>
</tr>
<tr>
<td>5.2 ± 0.3</td>
<td>15.00</td>
</tr>
<tr>
<td>7.8 ± 0.4</td>
<td>25.00</td>
</tr>
<tr>
<td>10.0 ± 0.5</td>
<td>35.00</td>
</tr>
<tr>
<td>12.5 ± 0.4</td>
<td>45.00</td>
</tr>
</tbody>
</table>

Table 1: Displacement vs. applied force for a spring.

Which shall we plot along the x-axis and which along the y-axis? A general rule is to plot the dependent variable along the y-axis. In other words, if the position of the indicator on the spring is to be expressed in terms of the force of the spring, then plot the indicator position on the y-axis. The scale units are chosen with consideration both for convenience and for the size of the diagram. We now make the assumption that the relation between the force $F$ applied to the spring and the extension $s$ of the spring is given by Hooke’s Law:

$$F = k(s - s_0),$$

written to include $s_0$, the position of the indicator when $F = 0$. We expect a linear relation between $F$ and $s$, and draw the best straight line through the points. If we now wish to determine the spring constant, $k$, we can do so easily from the slope of the line. The slope is obtained by taking two points on the straight line (NOT two experimental points, because the straight line represents the best fit to the experimental data and reflects all of the data, not just the two points chosen) say $(x_1, y_1)$ and $(x_2, y_2)$ and forming the quotient $(y_2 - y_1)/(x_2 - x_1)$. 

3
Measurement and Experimental Error

Whenever we measure a physical quantity – for example, a length measured by putting a ruler beside it or a mass measured by comparing it with known masses on a balance – there is some range of uncertainty in the result. No quantity is every measured with infinite precision. We can easily think of reasons for the uncertainty. Limitations of eyesight make it impossible to tell precisely where the end of the object falls on the ruler. If the length is read to be 10.25 inches, it could actually be 12.255 inches or 10.245 inches and look the same to our eye. Similarly, there will be a range within which mass may be added or subtracted from the balance pan without upsetting the appearance of the balance. Such effects will lead with equal probability to measurements that are too high or too low, and a given observer will probably get different values within the range of uncertainty for successive measurements. Errors of this sort are referred to as random. There may also be sources of systematic error which lead consistently to a number which is too large or too small. This could happen, for example, with a wooden ruler which had absorbed water on a humid day and increased its length or with a balance which had not been properly balanced when both pans were empty. Systematic errors are more difficult to detect than random errors because they do not produce different values for successive measurements. Nevertheless, the validity and usefulness of the experiment depends critically on the proper assessment of systematic errors.
Probability Distributions

Random errors may be treated by making a measurement several times and considering the distribution of results. Table 2 lists the results of a series of measurements of the length of an object. It is often convenient to display the distribution graphically as in Figure 2, where we have plotted on the horizontal axis the measured value of a length and on the vertical axis the number of times a particular value, or range of values within a given bin, was measured. This type of graph is called a histogram. Figure 2 illustrates a common property of such

<table>
<thead>
<tr>
<th>Length (cm)</th>
<th>Number of Times Measured</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.250</td>
<td>1</td>
</tr>
<tr>
<td>10.251</td>
<td>3</td>
</tr>
<tr>
<td>10.252</td>
<td>2</td>
</tr>
<tr>
<td>10.253</td>
<td>3</td>
</tr>
<tr>
<td>10.254</td>
<td>5</td>
</tr>
<tr>
<td>10.255</td>
<td>7</td>
</tr>
<tr>
<td>10.256</td>
<td>10</td>
</tr>
<tr>
<td>10.257</td>
<td>9</td>
</tr>
<tr>
<td>10.258</td>
<td>6</td>
</tr>
<tr>
<td>10.259</td>
<td>2</td>
</tr>
<tr>
<td>10.260</td>
<td>4</td>
</tr>
<tr>
<td>10.261</td>
<td>3</td>
</tr>
<tr>
<td>10.262</td>
<td>2</td>
</tr>
<tr>
<td>10.263</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Measured length of object.

a series of measurements. The values usually tend to cluster around a central value, in this case, around 10.256 cm, showing a “peak” of high probability for measuring values close to the center and showing fairly symmetrical “tails” of low probability for values from the center.

Given the data shown in Figure 2, what can we conclude about the true value of the length being measured? Can we say only that the length is somewhere between 10.250 cm and 10.263 cm? While it is probable that the true value lies somewhere within this range, it is most likely that it is somewhere near the center of the distribution. Our best estimate for the true value will be the average (or mean) of the distribution, which is defined as:

\[ \bar{x} = \frac{N_1 x_1 + N_2 x_2 + N_3 x_3 + \cdots}{N}, \]

where \( N_i \) is the number of times the value \( x_i \) occurs and where \( N \) is the total number of measurements, \( N = N_1 + N_2 + N_3 + \cdots \).

The most common way of assigning a size to the uncertainty associated with random error in a single measurement is to calculate the standard deviation \( \sigma_x \) of the distribution
Figure 2: Histogram of measured lengths.

from the formula

$$\sigma_x = \sqrt{\frac{N_1 (x_1 - \bar{x})^2 + N_2 (x_2 - \bar{x})^2 + N_3 (x_3 - \bar{x})^2 + \cdots}{N - 1}}$$  \hspace{1cm} (2)$$

or equivalently,

$$\sigma_x = \sqrt{\frac{N_1 x_1^2 + N_2 x_2^2 + N_3 x_3^2 \cdots - N \bar{x}^2}{N - 1}}$$ \hspace{1cm} (3)$$

The standard deviation is a measure of the uncertainty associated with a single measurement. A typical measurement can be expected to be within about a standard deviation from the mean value. Of course, some measurements have a smaller difference from the mean than $\sigma_x$ and some have a larger difference. Sometimes one uses the relative error ($\sigma_x/\bar{x}$, written as a percentage) to express the uncertainty in a single measurement. For the data in Table 2, $\bar{x} = 10.256$ cm, $\sigma_x = 0.003$ cm, and $\sigma_x/\bar{x} = 0.003 = 0.03\%$. Having estimated that, of all the estimates discussed, the mean value of the distribution is closest to the true value of the quantity, we can ask “How much confidence should we have in this estimate?” The answer is given by statistical theory: the average difference of the mean from the true value is of a size $\delta_x$ given by:

$$\delta_x = \frac{\sigma_x}{\sqrt{N}}.$$  \hspace{1cm} (4)$$

The number $\sigma_x$ is often called the standard deviation of the mean, but it is important to distinguish it from the standard deviation $\sigma_x$ of a single measurement. If we were to take more measurements ($i.e.$, increase $N$), $\sigma_x$ should not change much, but $\delta_x$ would become smaller. We often use this quantity $\delta_x$ as an error estimate to accompany a measured number, so that $x = 5.7 \pm 0.2$ cm means that the best estimate is $\bar{x} = 5.7$ cm and the uncertainty associated with this estimate is $\delta_x = 0.2$ cm.
Normal Distribution

According to statistical theory, a measured probability distribution such as Figure 2 can be approximated by a “normal distribution function”:

\[
\Delta N(x) = \frac{N}{\sqrt{2\pi}} \cdot \frac{\Delta x}{\sigma_x} \cdot \exp\left[ -\frac{(x - \bar{x})^2}{2\sigma_x^2} \right]
\]

where \( \Delta N(x) \) is the number of measurements between \( x \) and \( x + \Delta x \) and where \( N \) is the total number of measurements. \( \bar{x} \) and \( \sigma_x \) are the mean and the standard deviation. The normal distribution function is plotted in Figure 3.

Note again that there are long tails, indicating that a small fraction of the measurements will be much more than \( \sigma_x \) away from the mean value. For the normal distribution, 5 % of the measurements will be more than \( \pm 2\sigma_x \) away from \( \bar{x} \) while 68 % are within \( \pm \sigma_x \) of \( \bar{x} \). This well-known distribution is also sometimes called the “Gaussian” distribution, or the “Bell-curve”.

Subjectively Estimated Errors

Sometimes it is not practical to make the large number of measurements needed to calculate accurate estimates of the mean and standard deviation; instead, just one measurement is available to use as the estimate of the true value. In this case you must use your judgement
to estimate the uncertainty in this measurement. Several factors enter into this, some of which could be the following.

1. The uncertainty may come from how accurately you can read a measuring device. For example, the accuracy with which you could read a ruler would depend on how closely spaced its tick marks are, and on how steadily it could be held while the measurement was being made.

2. There may be fluctuations in the conditions under which an experiment was done which could affect the outcome of the experiment but which were beyond the control of the observer. These might include variations in the room temperature or the line voltage.

3. Your model of the physical situation may be somewhat unrealistic, so that no matter how precisely measurements are made, calculations using these measurements will not give an accurate result because the formulae aren’t a totally accurate model of reality. An example of this would be a calculation of the range of a projectile using a formula that neglects air resistance: the result might be fairly accurate for a bullet but not for a wad of paper, no matter how precisely you measure its initial velocity.

You must take the various factors into consideration and then exercise your best judgement in estimating the range within which you are confident that the measurement lies. The hope is that for random errors the estimate of the uncertainty that you make this way would be comparable to the value of $\sigma_x$ you would obtain if you made the measurement several times.

**Propagation of Errors**

Once the uncertainty in a measured quantity $x$ has been found, it is often necessary to calculate the consequential uncertainty in some other quantity which depends on $x$. Consider a function $f$ which depends on $x$ and/or $y$: $f = f(x)$ or $f = f(x, y)$. A series of measurements of $x$ and/or $y$ will yield $\bar{x}$, $\sigma_x$, $\bar{y}$, and $\sigma_y$. We will assume that the best estimate of $f$ is $\bar{f} = f(\bar{x})$ or $f(\bar{x}, \bar{y})$. (But note that $\bar{f}$ defined this way is not always the same as the mean of $f$ obtained from the individual values $f(x_i)$.)

Examples of the way to find $\sigma_f$, the standard deviation of $f$, are given below:

1. $f = x \pm y \quad \bar{f} = \bar{x} \pm \bar{y} \quad \sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}$

   The standard deviation for $f$ is obtained by adding the standard deviations for $x$ and $y$ “in quadrature”.

2. $f = ax \pm by \quad \bar{f} = a\bar{x} + b\bar{y} \quad \sigma_f = \sqrt{a^2\sigma_x^2 + b^2\sigma_y^2}$
3. \[ f = x \cdot y \quad \bar{f} = \bar{x} \cdot \bar{y} \quad \left[ \frac{\sigma_f}{f} \right] = \sqrt{\left( \frac{\sigma_x}{x} \right)^2 + \left( \frac{\sigma_y}{y} \right)^2} \]

The relative error for \( f \) is obtained by adding the relative errors for \( x \) and \( y \) “in quadrature”.

4. \[ f = \frac{x}{y} \quad \bar{f} = \frac{\bar{x}}{\bar{y}} \quad \left[ \frac{\sigma_f}{f} \right] = \sqrt{\left( \frac{\sigma_x}{x} \right)^2 + \left( \frac{\sigma_y}{y} \right)^2} \]

5. \[ f = Ax^{\pm b} \quad \bar{f} = A\bar{x}^{\pm b} \quad \left[ \frac{\sigma_f}{f} \right] = b \left[ \frac{\sigma_x}{x} \right] \]

where \( A \) and \( b \) are precisely-known constants and \( b \) is positive. The relative error for \( f \) is \( b \) times the relative error for \( x \).

The rules for calculating \( \sigma_f \) need some explanation. Let \( df \) be the small change in \( f \) which results from small changes \( dx, dy \) in the quantities \( x, y \). One can then make the identifications: \( df \rightarrow \sigma_f, \, dx \rightarrow \sigma_x, \, f \rightarrow \bar{f}, \, x \rightarrow \bar{x}, \) and \( y \rightarrow \bar{y} \). For example, let \( f = Ax^n \). Then,

\[ df = nA x^{n-1} dx; \text{ that is, } (df/f) = n(dx/x) \]

With the above identifications, we obtain \( (\sigma_f/\bar{f}) = n(\sigma_x/\bar{x}) \). As another example, let \( f = x + y \). Then \( df = dx + dy \). We might conclude that \( \sigma_f = \sigma_x + \sigma_y \). However, during half of the time, the deviations in \( x \) and \( y \) will be in opposite directions (as long as \( x \) and \( y \) are measured independently) so that one expects \( \sigma_f \) to be less than \( \sigma_x + \sigma_y \). A careful statistical analysis shows that \( \sigma_x \) and \( \sigma_y \) should be added “in quadrature”. All the rules above can easily be obtained by identifying differentials with standard deviations and by replacing addition or subtraction by addition “in quadrature”.