1. **Adjoint representation of** $SU(n)$

Let

$$V = \{n \times n \text{ Hermitian traceless matrices}\}.$$  

Define operators $\mathcal{D}(U)$ which act on $V$ according to

$$\mathcal{D}(U)A = UAU^\dagger$$  where $U \in SU(n)$ and $A \in V$. This defines the adjoint representation of $SU(n)$.

(i) Show that $V$ is a real vector space. That is, show that you can take linear combinations $\alpha_1 A_1 + \alpha_2 A_2$ where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $A_1, A_2 \in V$.

(ii) Show that $\mathcal{D}(U)$ is a linear operator on $V$.

(iii) Show that the operators $\mathcal{D}(U)$ form a representation of $SU(n)$.

(iv) Compute the dimension of this representation.

(v) Introduce an inner product on $V$ by

$$(A, B) = \text{Tr} (AB) \quad A, B \in V .$$

Show that, with respect to this inner product, the adjoint representation is unitary.

2. **Parity in a magnetic field**

Consider the Hamiltonian for a particle in a uniform constant magnetic field. Is the Hamiltonian invariant under parity?

3. **Band structure**

Consider a particle moving in a one-dimensional periodic array of $\delta$-function potentials:

$$H = \frac{p^2}{2m} + V_0 \sum_{n=-\infty}^{\infty} \delta(x + na) .$$

This is known as the Kronig-Penney model. For simplicity take $V_0 > 0$. 
In this problem you’ll determine the energy levels of this Hamiltonian, using the fact that the energy eigenfunctions can be written in the form

\[ \psi(x) = e^{i \theta x/a} u(x) \quad -\pi \leq \theta \leq \pi \]

where \( u(x) \) is simply periodic: \( u(x + a) = u(x) \).

(i) The wavefunction

\[ \psi(x) = Ae^{ikx} + Be^{-ikx} \quad 0 \leq x \leq a \]

is potentially an eigenstate with energy \( E = \hbar^2 k^2 / 2m \). Without loss of generality we can take \( k > 0 \). By matching at the \( \delta \)-functions, show that the allowed values of \( k \) satisfy

\[ \cos \theta = \cos ka + \frac{mV_0 \sin ka}{\hbar^2 k} . \]

Careful – you must use the correct periodicity properties to obtain the wavefunction outside the range \( 0 \leq x \leq a \)!

For any \( \theta \) this equation has an infinite number of solutions. Call these solutions \( k_1, k_2, \ldots \) and denote the corresponding states \( |\theta, n\rangle \), \( n = 1, 2, \ldots \)

(ii) Qualitatively plot the allowed energy levels vs. \( \theta \). Note that \( n \) labels the different allowed “energy bands.”

4. Parity and band structure

Consider a particle moving in a periodic potential \( V(x) = V(x + a) \). Assume that \( V(x) = V(-x) \), so the Hamiltonian is invariant under parity. The Kronig-Penney model is an example of such a Hamiltonian. Label the energy eigenstates \( |\theta, n\rangle \), where \( \theta \) determines the eigenvalue of the translation operator, and \( n \) labels the different energy bands: \( T_a |\theta, n\rangle = e^{-i \theta} |\theta, n\rangle \).

(i) Assuming that the different energy bands do not overlap, use the parity operator to show that the states \( |\theta, n\rangle \) and \( |-\theta, n\rangle \) must be degenerate in energy.

(ii) When \( \theta = \pi \) the two states \( |\theta, n\rangle \) and \( |-\theta, n\rangle \) located at the edge of an energy band could in fact be the same state (up to a phase). Assume that this is the case. That is, assume that there is no degeneracy at the edge of an energy band. Show that the states at the edge of an energy band are parity eigenstates.

5. Degenerate fermions

Suppose \( N \) non-interacting fermions are placed in a 3-dimensional harmonic oscillator potential \( V(x) = \frac{1}{2} m \omega^2 |x|^2 \). Assume that \( N \gg 1 \), and determine the ground state energy of the system.